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EDITED BY
T. A. A. BROADBENT, M.A.

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T. A. A. BROADBENT, M.A.
62 Coleraine Road, Blackheath, London, S.E. 3

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Vol. XX. December, 1936. No. 241

WILLIAM FLEETWOOD SHEPPARD

Dr. W. F. Sheppard, late Assistant Secretary of the Board of Education, was the Senior Wrangler of 1886. As a Fellow of Trinity, he resided in Cambridge for some years after taking his degree ; afterwards he was called to the Bar, and later became a member of the staff of the Board of Education, where his knowledge of statistics proved very valuable. It was in the study of statistical problems that he made his most valuable contributions to mathematical knowledge, and his name is assured of a permanent place on the list of those who have added to the scope and power of statistical methods.

Dr. Sheppard joined the Association in 1906 ; he served on the Council as an ordinary member from 1920 to 1925, and as a co-opted representative of the London Branch for 1926 and 1927. As President of the Association in 1928 and 1929, he was able to share two of his keenest mathematical interests with members through his presidential addresses, " Variety of Method in the Teaching of Arithmetic " and " Mathematics for the Study of Frequency Statistics ". He remained on the Council as a Vice-President ; a wonderfully regular attendant at Council meetings until his health began to fail, his shrewd judgment was of great value. The London Branch, of which he was President in 1926, found in him an active and powerful helper.

So varied were Dr. Sheppard's interests that our Association is but one of many which now deplore a severe loss.

EDDINGTON'S PROBABILITY PROBLEM.

By H. Wallis Chapman.

In his *New Pathways in Science* (p. 121) Sir Arthur Eddington propounds the following problem :

" If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D is speaking the truth ? " and he has dealt with it further in the *Mathematical Gazette* for October 1935.

This problem has excited some attention, and has been discussed by Dr. Dingle and Dr. Sterne in *Nature* (vol. 135, p. 453 and p. 1073; vol. 136, p. 301 and p. 423). I therefore thought that a thorough examination of it might be of interest, and such an examination follows. The problem is in itself trivial, but as an example of reasoning on probability, and particularly of the power of the notation set forth by J. M. Keynes in his *Treatise on Probability* to remove ambiguities and reveal unrecognized assumptions, the investigation may have some value.

1. I will begin with a few pieces of notation.

h will be used, as by Keynes, to denote that background of knowledge which forms part of the supposition in all our probabilities.

The later letters of the alphabet, x, y, z, will be used to denote statements as to which h gives no further information than is implied by their being statements.

" A makes a statement x " will be denoted by (ax) ; " A says that B says x " by $(a(bx)$, and so forth.

As Keynes points out (*loc. cit.* p. 183), the phrase " the credibility of a witness A " is ambiguous ; it may mean the probability that any statement he makes on a proposed subject is true, or that any statement of his taken at random is true. These will be denoted by $(ax)/x \,.\, h$ and $x/(ax) \,.\, h$, and called the veracity and credibility of A respectively.

Another form, which is neutral between the two, is

$$\{(ax) \,.\, x + (a\bar{x}) \,.\, \bar{x}\}/h \,;$$

but this seems to have little or no possible application.

We shall find, however, that in certain instances, especially in regard to the statements made by D, it is unnecessary to distinguish these meanings. With regard to the statements of A, B and C, on the other hand, we shall find that a constant credibility leads to contradictions.

Another point which must be noticed is as follows :

$x/(ax) \,.\, h$ is the probability that an otherwise undescribed statement made by A is true ; we are tempted to suppose that we may substitute some other symbol, denoting a constant or a variable of limited range for x, and write, for instance, $(bx)/(a(bx) \,.\, h$; but, as

Keynes has pointed out (*loc. cit.* p. 58), this involves a risk of false assumptions. We must write for the probability required

$$x/(ax) . x=(by):h,$$

and we see at once that we cannot assume the probability to be unaltered, for $x=(by)$ is a new piece of information concerning x, and we cannot assume without enquiry that it is irrelevant.

Dots will be used in the now recognized way for combining propositions.

2. The relations of A's credibility and veracity as above defined to the neutral form $\dfrac{x . (ax)+\bar{x} . (a\bar{x})}{h}$ can be set forth thus:

If A's credibility is constant,

$$x/(ax) . h=\bar{x}/(a\bar{x}) . h; \qquad 2.1$$

and if we denote " A is silent as respects x " by $a_s(x)$,

$$\{(ax)+(a\bar{x})+a_s(x)\}/h=1. \qquad 2.2$$

Then $\{x . (ax)+\bar{x} . (a\bar{x})\}/h=x/(ax) . h\times(ax)/h+\bar{x}/(a\bar{x})h\times(a\bar{x})/h$

$$=x/ax . h\times\{(ax)/h+(a\bar{x})h\} \text{ from } 2.1$$

$$=x/ax . h\times\{1-a_s(x)/h\}; \qquad 2.3$$

if $\dfrac{a_s(x)}{h}=0$, or it is known that A has spoken,

$$\frac{x . (ax)+\bar{x} . (a\bar{x})}{h}=\frac{x}{(ax) . h}, \qquad 2.3a$$

or the neutral form is equal to the credibility.

If the veracity is constant,

$$(ax)/x . h=(a\bar{x})/\bar{x} . h, \qquad 2.4$$

and

$$\frac{x . (ax)+\bar{x} . (a\bar{x})}{h}=\frac{(ax)}{x . h}\times\frac{x}{h}+\frac{(a\bar{x})}{\bar{x} . h}\times\frac{\bar{x}}{h}$$

$$=\frac{(ax)}{x . h}\times\left(\frac{x}{h}+\frac{\bar{x}}{h}\right) \text{ by } 2.4$$

$$=\frac{(ax)}{x . h}, \qquad 2.5$$

or the neutral form is equal to the veracity whether A speaks or not.

If the veracity and credibility are both constant we have, by 2.3 and 2.5,

$$\frac{x}{(ax) . h}\left(1-\frac{a_s(x)}{h}\right)=\frac{(ax)}{x . h}; \qquad 2.6$$

so that the veracity and credibility cannot be equal unless it is known that A has spoken.

If this condition is satisfied it is easy to show (Keynes, p. 183) that

$$x/h=\bar{x}/h=\tfrac{1}{2}. \qquad 2.7$$

It may plausibly be held, in spite of Keynes' arguments to the contrary, that if x is absolutely unrestricted this condition is fulfilled, as there is a one-one relation of contradiction between true and false propositions. But if x is in any way restricted this argument fails, and the above reasoning shows that the veracity and credibility of a witness cannot be the same in respect of any class of assertions defined by a function $f(x)$ unless

$$x/f(x) \,.\, h = \bar{x}/f(x) \,.\, h = \tfrac{1}{2}. \qquad \text{2.8}$$

[The veracity and credibility could be made equal by appropriate assumptions about $a_s(x)/f(x) \,.\, h$ and $a_s(x)/x \,.\, f(x) \,.\, h$, but this seems of little interest.]

3. Before proceeding to the main problem I will deal with a simplification discussed by Dr. Dingle and Dr. Sterne (*Nature, loc. cit.*), but will generalize it, as I shall the main problem, by taking any values for the probabilities involved. The simplified problem is: " C declares that D's statement is false; what is the probability that it is true?"

In D's case I will take the neutral form referred to above and write:

$$x(dx) + \bar{x}(d\bar{x}) = D, \qquad \text{3.1}$$

$$D/h = \delta. \qquad \text{3.2}$$

4. We will first assume that C's credibility is constant or that

$$x/(cx) \,.\, h = \gamma, \qquad \text{4.1}$$

and denote " D makes no assertion about x " by $d_s(x)$.

Then

$$\overline{D} = \underline{D} + d_s(x) \qquad \text{4.2}$$

where

$$\underline{D} = x \,.\, (d\bar{x}) + \bar{x} \,.\, (dx). \qquad \text{4.3}$$

Then the probability required is

$$\frac{D}{(cx) \,.\, x = \underline{D} : h} = 1 - \frac{\overline{D}}{(cx) \,.\, x = \underline{D} : h}$$

$$= 1 - \frac{\underline{D}}{(cx) \,.\, x = \underline{D} : h} - \frac{d_s(x)}{(cx) \,.\, x = \underline{D} : h}. \qquad \text{4.4}$$

But our data give us no information about

$$\frac{d_s(x)}{(cx) \,.\, x = \underline{D} : h},$$

and it does not seem reasonable that they should; a statistical average of the number of times C lies may be reasonable, but probabilities of the various lies which he tells hardly seem so.

We therefore assume that D speaks, and obtain

$$\frac{D}{(cx) \,.\, x = \underline{D} : h} = 1 - \frac{\underline{D}}{(cx) \,.\, x = \underline{D} : h} = 1 - \gamma, \qquad \text{4.5}$$

assuming that $x = \underline{D}$ is irrelevant to $\dfrac{x}{(cx)h}$.

This is Dr. Dingle's solution, but the fact of irrelevance is not given in the data, and can be shown to be inconsistent with them. For, assuming that C speaks,

$$D/h = \frac{D:(cx).x=D}{h} + \frac{D:(cx).x=\underline{D}}{h}$$

$$= \frac{D}{(cx).x=D:h} \times \frac{(cx).x=D}{h} + \frac{D}{(cx).x=\underline{D}:h} \times \frac{(cx).x=\underline{D}}{h},$$

whence $$\frac{(cx).x=D}{h} = \frac{\gamma+\delta-1}{2\gamma-1}; \quad \text{.........................4.6}$$

and in our particular problem $\gamma=\delta=\frac{1}{3}$, which gives

$$\frac{(cx).x=D}{h}=1,$$

or C always states that D tells the truth, which is contrary to the data.

It may be of interest to put this proof of inconsistency into the notation of the frequency theory. It will be necessary to abandon the neutral form and take D's credibility to be constant. We will also abandon the assumption that C speaks. [The proof above can easily be modified in the same way.]

Then on N occasions let C say D speaks true pN times,
" " " " " false qN times,
" " " C keep silent $(1-p-q)N$ times.

Then of the pN times D speaks truly $p\gamma N$ times,
" " " " falsely $p(1-\gamma)N$ times,
" qN times D " truly $q(1-\gamma)N$ times,
" " " " falsely $q\gamma N$ times,

and, assuming that the fact of C's silence is irrelevant to D's credibility:

Of the $(1-p-q)N$ times D speaks truly $(1-p-q)\delta N$ times,
and falsely $(1-p-q)(1-\delta)N$ times;

and on the whole N occasions D speaks truly δN times;

$$\therefore\ \delta = p\gamma + q(1-\gamma) + (1-p-q)\delta,$$

or $$0 = p(\gamma-\delta) + q(1-\gamma-\delta),$$

whence if $\gamma=\delta=\frac{1}{3}$, $q=0$; which agrees with our former result.

5. We will therefore take C's veracity to be constant, or

$$(cx)/x.h=\gamma. \quad \text{..............................5.1}$$

We must also assume that D and $\underline{D}$ are fair samples of C's statements, so

$$\{(cx).x=D\}/D.h = \{(cx).x=\underline{D}\}/\underline{D}.h=\gamma. \quad \text{............5.2}$$

Then

$$\frac{D:(cx)\,.\,x=\underline{D}}{h}=\frac{D}{(cx)\,.\,x=\underline{D}:h}\times\frac{(cx)\,.\,x=\underline{D}}{h}$$

$$=\frac{D}{(cx)\,.\,x=\underline{D}:h}\left\{\frac{(cx)\,.\,x=\underline{D}:\underline{D}}{h}+\frac{(cx)x=\underline{D}:\underline{\overline{D}}}{h}\right\}$$

$$=\frac{D}{(cx)\,.\,x=\underline{D}\,.\,h}\left\{\frac{(cx)\,.\,x=\underline{D}}{\underline{D}\,.\,h}\times\frac{\underline{D}}{h}\right.$$

$$\left.+\frac{(cx)\,.\,x=\underline{D}}{\underline{\overline{D}}\,.\,h}\times\frac{\underline{\overline{D}}}{h}\right\}.$$

Also
$$\frac{D:(cx)\,.\,x=\underline{D}}{h}=\frac{(cx)\,.\,(x=\underline{D})}{D\,.\,h}\times\frac{D}{h},$$

and we meet with the same difficulty as before; for though we are given $\frac{(cx)\,.\,\overline{x=D}}{Dh}$, this is not the same thing as $\frac{(cx)\,.\,x=\underline{D}}{D\,.\,h}$, and involves unreasonable demands on our data.

We will therefore assume that D speaks so that $\underline{D}=\overline{D}$, and continue:

$$\frac{D:(cx)\,.\,x=\overline{D}}{h}=\frac{D}{(cx)\,.\,x=\overline{D}:h}\left\{\frac{(cx)\,.\,x=\overline{D}}{\overline{D}\,.\,h}\times\frac{\overline{D}}{h}+\frac{(cx)\,.\,x=\overline{D}}{D\,.\,h}\times\frac{D}{h}\right\}$$

$$=\frac{D}{(cx)\,.\,x=\overline{D}:h}\{\gamma(1-\delta)+(1-\gamma-\gamma_s)\delta\},^{*} \quad \text{.........5.3}$$

where γ_s is the probability that C says nothing to the point.

Also

$$\frac{D:(cx)\,.\,x=\overline{D}}{h}=\frac{(cx)\,.\,x=\overline{D}}{D\,.\,h}\times\frac{D}{h}$$

$$=(1-\gamma-\gamma_s)\delta. \quad \text{.......................5.4}$$

So that the probability required is

$$\frac{D}{(cx)\,.\,x=\overline{D}:h}=\frac{(1-\gamma-\gamma_s)\delta}{\gamma(1-\delta)+(1-\gamma-\gamma_s)\delta}. \quad \text{................5.5}$$

If $\gamma_s=0$ and $\gamma=\delta$ this becomes 1/2, which is Dr. Sterne's solution.

The same result is easily obtained by the method of exclusion.

* This value for $(cx)\,.\,x=\overline{D}/D\,.\,h$ implies that γ is not the proportion of C's statements which are true, but of those which are true and relevant; if we suppose that our data give the proportion of C's statements which are true, whether relevant or not, and that C always says something, we can express this by writing

$$(cx)\,.\,x=D/D\,.\,h=(1-\gamma_s)\gamma \quad \text{and} \quad (cx)\,.\,x=\overline{D}/D\,.\,h=(1-\gamma_s)(1-\gamma).$$

The general reasoning will be unaltered, but of course the numerical examples will be affected if we use the same value of γ in both cases.

The fact that the result contains γ_s, although the supposal "C declares that D lies" implies that C has spoken, may appear paradoxical, but it is easily seen to be correct, for the value of γ_s limits the field from which the instances to be compared are taken.

6. We will now return to the original problem and represent the statements involved thus:

$c = :(cx).x = \overline{D}$, $\underline{c} = :(cx).x = D$, $c_s = C$ says nothing to the point.

$b = :(bx).\overline{x = c} : = \vdots (bx) : x = .\underline{c} \vee c_s :$, $\underline{b} = :(bx).x = c$,

$b_s = B$ says nothing to the point.

$a = :(ax).x = b$, $\underline{a} = \bar{a} = :(ax).\overline{x = b} : = \vdots (ax) : x = \underline{b} \vee b_s$.

We have here assumed that D speaks and that A speaks to the point.

Then
$$\begin{aligned} c/h &= c.D/h + c.\overline{D}/h \\ &= c/D.h \times D/h + c/\overline{D}.h \times \overline{D}/h \\ &= (1 - \gamma - \gamma_s)\delta + \gamma(1 - \delta) \\ &= \tfrac{1}{2}\{1 - (2\gamma - 1)(2\delta - 1)\} - \gamma_s\delta, \quad \text{......} 6.1 \end{aligned}$$

assuming that $c_s/D.h = c_s/h = \gamma_s$

and $c/D.h = \gamma$,

or that "D speaks truth" is a fair sample of C's statements; without such an assumption we cannot of course advance a step.

$$\begin{aligned} \underline{c}/h &= \underline{c}.D/h + \underline{c}.\overline{D}/h \\ &= \underline{c}/D.h \times D/h + \underline{c}/\overline{D}.h \times \overline{D}/h \\ &= \gamma\delta + (1 - \gamma - \gamma_s)(1 - \delta) \\ &= \tfrac{1}{2}\{1 + (2\gamma - 1)(2\delta - 1)\} - \gamma_s(1 - \delta); \quad \text{......} 6.2 \end{aligned}$$

$$\begin{aligned} b/h &= b.c/h + b.\underline{c}/h + b.c_s/h \\ &= b/c.h \times c/h + b/\underline{c}.h \times \underline{c}/h + b/c_s.h \times c_s/h \\ &= (1 - \beta - \beta_s)[\tfrac{1}{2}\{1 - (2\gamma - 1)(2\delta - 1)\} - \gamma_s\delta] \\ &\quad + \beta[\tfrac{1}{2}\{1 + (2\gamma - 1)(2\delta - 1)\} - \gamma_s(1 - \delta)] \\ &\quad + \beta\gamma_s, \text{ from 6.1 and 6.2,} \\ &= \tfrac{1}{2}\{1 + (2\beta - 1)(2\gamma - 1)(2\delta - 1)\} \\ &\quad + \tfrac{1}{2}\beta_s\{(2\gamma - 1)(2\delta - 1) - 1\} \\ &\quad + \gamma_s\delta(2\beta - 1) + \beta_s\gamma_s\delta. \quad \text{......} 6.3 \end{aligned}$$

Similarly

$$\begin{aligned} \underline{b}/h = \tfrac{1}{2}\{1 - (2\beta - 1)(2\gamma - 1)(2\delta - 1)\} - \tfrac{1}{2}\beta_s\{1 + (2\gamma - 1)(2\delta - 1)\} \\ - \gamma_s\delta(2\beta - 1) - \beta_s\gamma_s\delta. \quad \text{......} 6.4 \end{aligned}$$

$$\begin{aligned} a/h &= a.b/h + a.\underline{b}/h + a.b_s/h \\ &= a/b.h \times b/h + a/\underline{b}.h \times \underline{b}/h + a/b_s.h \times b_s/h \end{aligned}$$

$$\begin{aligned}
&=\alpha[\tfrac{1}{2}\{1+(2\beta-1)(2\gamma-1)(2\delta-1)\}+\tfrac{1}{2}\beta_s\{(2\gamma-1)(2\delta-1)-1\}\\
&\qquad+\gamma_s\delta(2\beta-1)+\beta_s\gamma_s]\\
&\qquad+(1-\alpha)[\tfrac{1}{2}\{1-(2\beta-1)(2\gamma-1)(2\delta-1)\}\\
&\qquad-\tfrac{1}{2}\beta_s\{(2\gamma-1)(2\delta-1)+1\}\\
&\qquad-\gamma_s\delta(2\beta-1)-\beta_s\gamma_s],\text{ from 6.3 and 6.4,}\\
&\qquad+(1-\alpha)\beta_s\\
&=\tfrac{1}{2}\{1+(2\alpha-1)(2\beta-1)(2\gamma-1)(2\delta-1)\}\\
&\qquad+\tfrac{1}{2}\beta_s(2\alpha-1)\{(2\gamma-1)(2\delta-1)-1\}\\
&\qquad+\gamma_s\delta(2\alpha-1)(2\beta-1)+\beta_s\gamma_s(2\alpha-1). \quad\ldots\ldots 6.5
\end{aligned}$$

Again,

$$\begin{aligned}
b/D\,.\,h&=b\,.\,c/D\,.\,h+b\,.\,\underline{c}/D\,.\,h+b\,.\,c_s/D\,.\,h\\
&=b/c\,.\,D\,.\,h\times c/D\,.\,h+b/\underline{c}\,.\,D\,.\,h\times \underline{c}/D\,.\,h+b/c_s\,.\,D\,.\,h\times c_s/D\,.\,h\\
&=(1-\beta-\beta_s)(1-\gamma-\gamma_s)+\beta\gamma+\beta\gamma_s.
\end{aligned}$$

Assuming that the probability that B will report C correctly is not affected by the truth or falsehood of D's statement; hence

$$b/D\,.\,h=\tfrac{1}{2}\{(2\beta-1)(2\gamma-1)+1\}-\beta_s(1-\gamma)+\gamma_s(2\beta-1)+\beta_s\gamma_s. \quad 6.6$$

Similarly

$$\underline{b}/D\,.\,h=\tfrac{1}{2}\{1-(2\beta-1)(2\gamma-1)\}-\beta_s\gamma-\gamma_s(2\beta-1)-\beta_s\gamma_s \quad\ldots\ldots 6.7$$

and

$$\begin{aligned}
a/Dh&=a\,.\,b/D\,.\,h+a\,.\,\underline{b}/D\,.\,h+a\,.\,b_s/D\,.\,h\\
&=a/b\,.\,D\,.\,h\times b/D\,.\,h+a/\underline{b}\,.\,D\,.\,h\times \underline{b}/D\,.\,h\\
&\qquad+a/b_s\,.\,D\,.\,h\times b_s/D\,.\,h\\
&=\alpha[\{\tfrac{1}{2}(2\beta-1)(2\gamma-1)+1\}-\beta_s(1-\gamma)+\gamma_s(2\beta-1)+\beta_s\gamma_s]\\
&\qquad+(1-\alpha)[\tfrac{1}{2}\{1-(2\beta-1)(2\gamma-1)\}-\beta_s\gamma-\gamma_s(2\beta-1)\\
&\qquad-\beta_s\gamma_s]+(1-\alpha)\beta_s\\
&=\tfrac{1}{2}\{1+(2\alpha-1)(2\beta-1)(2\gamma-1)\}-\beta_s(1-\gamma)(2\alpha-1)\\
&\qquad+\gamma_s(2\alpha-1)(2\beta-1)+\beta_s\gamma_s(2\alpha-1). \quad\ldots\ldots 6.8
\end{aligned}$$

And the required probability is

$$\begin{aligned}
D/a\,.\,h&=\frac{a/D\,.\,h\times D/h}{a/h}\\
&=\frac{\begin{array}{l}\delta[\tfrac{1}{2}\{1+(2\alpha-1)(2\beta-1)(2\gamma-1)\}-\beta_s(1-\gamma)(2\alpha-1)\\ \quad+\gamma_s(2\alpha-1)(2\beta-1)+\beta_s\gamma_s(2\alpha-1)]\end{array}}{\begin{array}{l}\tfrac{1}{2}\{1+(2\alpha-1)(2\beta-1)(2\gamma-1)(2\delta-1)\}\\ \quad+\tfrac{1}{2}\beta_s(2\alpha-1)\{(2\gamma-1)(2\delta-1)-1\}\end{array}}\\
&\qquad+\gamma_s\delta(2\alpha-1)(2\beta-1)+(\beta_s\gamma_s)(2\alpha-1)\delta, \quad\ldots\ldots 6.9
\end{aligned}$$

from 6.5 and 6.8.

7. In the particular case we put $\alpha=\beta=\gamma=\delta=\frac{1}{3}$, and have the required probability

$$\frac{13+\ 6\beta_s+3\gamma_s-9\beta_s\gamma_s}{41+123\beta_s+\gamma_s-9\beta_s\gamma_s}. \quad\ldots\ldots 7.1$$

This obviously cannot be made independent of β_s and γ_s, so that

without some assumption as to these quantities the problem cannot be solved.

The simplest assumption seems to be $\beta_s=\gamma_s=0$, or it is known that the witnesses have said something to the point, which gives the value 13/41.

Eddington's solution attempts to be independent of β_s and γ_s, and is 25/71 ; and it can easily be shown that the expression 6.10 cannot be made to take this value by giving any possible values to β_s and γ_s.

The cause of this discrepancy appears to be as follows :

When Eddington is making his table he neglects the difference between the cases he denotes by *TTTL* and *TTLL* on the ground that we do not know that *C* has said anything to the point at all, so that both are possible, whereas if he had done so *TTTL* would be impossible and *TTLL* possible. But this neglects the fact that, if we are given *TTT*, *C* may have something excluding the possibility of *D* lying, but he cannot have said anything inconsistent with *D* speaking the truth, while if we are given *TTL* the converse is the case. To neglect this is to make an assumption about γ_s, and it is easily shown that the assumption is that in these cases $\gamma_s=1-2\gamma$. If, however, we take the cases beginning *TL* and treat them in the same way, we find the necessary assumption to be $\gamma_s=0$, and the two are not compatible unless $\gamma=\frac{1}{2}$.

[*Note.*—It might be thought that, since "*A* truths and *B* lies" implies that *C* speaks, the value of γ_s could make no difference to the *TL* ... cases, but this is not true ; the case is precisely similar to that discussed in § 5, where it appeared that the *supposal* that *C* spoke did not exclude the effect of the *initial probability* that he was silent. A reconciling hypothesis expressing the dependence of γ_s on β and γ could probably be found, but is not easy to set forth.]

8. If we assume that all the witnesses say something to the point, the relation between the solution just given and that given by Dr. Dingle in *Nature* (Sept. 14, 1935) becomes clear. For in this case "*A* affirms that *B* denies that *C* affirms that *D* lies", or in his notation

$$A \rightarrow B \leftarrow C \rightarrow D -$$

is equivalent to the following three :

$$A \rightarrow B \rightarrow C \rightarrow D +,$$

$$A \rightarrow B \rightarrow C \leftarrow D -,$$

$$A \rightarrow B \leftarrow C \leftarrow D +,$$

and on collecting these forms in his table we get my value of 13/41.

9. In fact, we can see that, however many witnesses there may be, if all are supposed to speak to the point, the various combinations of affirmations and denials reduce to two cases only.

For if we call the original witness *D* as before, and the others in order $A_1 \ldots A_n$, "A_p denies that A_{p-1} says" is equivalent to "A_p

says that A_{p-1} denies", and "A_p denies that A_{p-1} denies" is equivalent to "A_p says that A_{p-1} says", and by means of these transformations we can reduce every case either to

"A_n says that A_{n-1} says ... that A_1 says that D speaks true",

or to

"A_n says that A_{n-1} says ... that A_1 says that D lies".

10. I will now give a solution of the problem for any number of witnesses on this assumption.

Denote

"A_p says that A_{p-1} says ... that A_1 says that D speaks true"

by a_p, and let A_p's veracity $=\alpha_p$.

Then

$$\begin{aligned} D \,.\, a_p/h &= D/a_p \,.\, h \,.\times a_p/h \\ &= D/a_p \,.\, h \times (a_p \,.\, a_{p-1}/h + a_p \,.\, \bar{a}_{p-1}/h) \\ &= D/a_p \,.\, h \times \left(\frac{a_p}{a_{p-1} \,.\, h} \times \frac{a_{p-1}}{h} + \frac{a_p}{\bar{a}_{p-1} \,.\, h} \times \frac{\bar{a}_{p-1}}{h}\right) \\ &= D/a_p \,.\, h \times \left\{\alpha_p \times \frac{a_{p-1}}{h} + (1-\alpha_p) \times \frac{\bar{a}_{p-1}}{h}\right\}. \quad \text{..............10.1} \end{aligned}$$

Also $$\begin{aligned} D \,.\, a_p/h &= a_p/D \,.\, h \times D/h \\ &= a_p/D \,.\, h \times \delta \,; \quad \text{..............10.2} \end{aligned}$$

and $$\begin{aligned} a_p/D \,.\, h &= a_p \,.\, a_{p-1}/D \,.\, h + a_p \,.\, \bar{a}_{p-1}/D \,.\, h \\ &= a_p/a_{p-1} \,.\, D \,.\, h \times a_{p-1}/D \,.\, h \\ &\qquad + a_p/\bar{a}_{p-1} \,.\, D \,.\, h \times \bar{a}_{p-1}/D \,.\, h \\ &= \alpha_p \times a_{p-1}/D \,.\, h + (1-\alpha_p) \times \bar{a}_{p-1}/D \,.\, h, \quad \text{..........10.3} \end{aligned}$$

assuming, as we did before, that the actual truth or falsehood of D's statement is irrelevant to the truth of the secondary witnesses' reports.

Similarly

$$\bar{a}_p/D \,.\, h = \alpha_p \times \bar{a}_{p-1}/D \,.\, h + (1-\alpha_p) \times a_{p-1}/D \,.\, h. \quad \text{........10.4}$$

Subtracting 10.4 from 10.3 we have

$$a_p/D \,.\, h - \bar{a}_p/D \,.\, h = (2\alpha_p - 1)(a_{p-1}/D - \bar{a}_{p-1}/D \,.\, h).$$

Whence immediately

$$\begin{aligned} a_p/D \,.\, h - \bar{a}_p/D \,.\, h &= (2\alpha_p - 1)(2\alpha_{p-1} - 1)(2\alpha_2 - 1)(a_1/D \,.\, h - \bar{a}_1/D \,.\, h) \\ &= (2\alpha_p - 1) \dots (2\alpha_1 - 1). \quad \text{..............10.5} \end{aligned}$$

And as

$$a_p/D \,.\, h + \bar{a}_p/D \,.\, h = 1,$$

$$\left.\begin{aligned} a_p/D \,.\, h &= \tfrac{1}{2}\{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \dots (2\alpha_1 - 1)\} \\ \bar{a}_p/D \,.\, h &= \tfrac{1}{2}\{1 - (2\alpha_p - 1)(2\alpha_{p-1} - 1) \dots (2\alpha_1 - 1)\} \end{aligned}\right\}. \quad \text{.......10.6}$$

Similarly we have

$$a_p/h - \bar{a}_p/h = (2\alpha_p - 1)(2\alpha_{p-1} - 1) \ldots \left(\frac{a_1}{h} - \frac{\bar{a}_1}{p}\right),$$

and $\dfrac{a_1}{h} - \dfrac{\bar{a}_1}{h}$ is easily shown to be equal to $(2\alpha_1 - 1)(2\delta - 1)$.

Hence

$$a_p/h - \bar{a}_p/h = (2\alpha_p - 1) \ldots (2\alpha_1 - 1)(2\delta - 1),$$

and

$$\left.\begin{aligned} a_p/h &= \tfrac{1}{2}\{1 + (2\alpha_p - 1) \ldots (2\alpha_1 - 1)(2\delta - 1)\} \\ \bar{a}_p/h &= \tfrac{1}{2}\{1 - (2\alpha_p - 1) \ldots (2\alpha_1 - 1)(2\delta - 1)\} \end{aligned}\right\}. \quad \text{.........10.7}$$

From 10.1, 10.2, 10.6 and 10.7 we have

$$\left.\begin{aligned} \frac{D}{a_p \,.\, h} &= \frac{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \ldots (2\alpha_1 - 1)}{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \ldots (2\alpha_1 - 1)(2\delta - 1)}\,\delta \\ &\text{and} \\ \frac{D}{\bar{a}_p \,.\, h} &= \frac{1 - (2\alpha_p - 1)(2\alpha_{p-1} - 1) \ldots (2\alpha_1 - 1)}{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \ldots (2\alpha_1 - 1)(2\delta - 1)}\,\delta \end{aligned}\right\}. \quad \text{......10.8}$$

Two points may be noted.

As our witnesses increase in number the result always approaches more and more closely to D's veracity δ.

If any one of the witnesses is as likely to speak truth as falsehood the required probability is equal to D's veracity, whatever the number of witnesses may be.

11. I will recapitulate the assumptions that have been made.

(1) The veracity of a witness is the probability that if a statement is true he will make it when occasion arises; and that if the statement relates to a statement by a second witness the actual truth or falsehood of this statement is irrelevant to the probability in question.

(2) That all witnesses are known to have spoken.

If these conditions are satisfied the probability required is that given in the last section; if they are not, the data are insufficient or inconsistent.

12. These assumptions give an air of unreality to the whole problem, for though logically consistent they are psychologically absurd. I have therefore devised a mechanical equivalent of the problem in which psychological considerations do not arise.

" D speaks truth " will be represented by a round block and " D lies " by a square block; these blocks may be supposed to be taken at random from a store containing twice as many square blocks as round.

" C says that D speaks truth " will be represented by a round block, and " C says that D lies " by a square block; each of these blocks has a hole in it, into which one of the D blocks will fit; a true statement will be represented by a block of the same shape as the hole in it, and a falsehood by a block of a different shape from

says that A_{p-1} denies", and "A_p denies that A_{p-1} denies" is equivalent to "A_p says that A_{p-1} says", and by means of these transformations we can reduce every case either to

"A_n says that A_{n-1} says ... that A_1 says that D speaks true", or to

"A_n says that A_{n-1} says ... that A_1 says that D lies".

10. I will now give a solution of the problem for any number of witnesses on this assumption.

Denote

"A_p says that A_{p-1} says ... that A_1 says that D speaks true" by a_p, and let A_p's veracity$=\alpha_p$.

Then

$$\begin{aligned} D\,.\,a_p/h &= D/a_p\,.\,h\,.\times a_p/h \\ &= D/a_p\,.\,h \times (a_p\,.\,a_{p-1}/h + a_p\,.\,\bar{a}_{p-1}/h) \\ &= D/a_p\,.\,h \times \left(\frac{a_p}{a_{p-1}\,.\,h} \times \frac{a_{p-1}}{h} + \frac{a_p}{\bar{a}_{p-1}\,.\,h} \times \frac{\bar{a}_{p-1}}{h}\right) \\ &= D/a_p\,.\,h \times \left\{\alpha_p \times \frac{a_{p-1}}{h} + (1-\alpha_p) \times \frac{\bar{a}_{p-1}}{h}\right\}. \quad \text{.............. 10.1} \end{aligned}$$

Also $$\begin{aligned} D\,.\,a_p/h &= a_p/D\,.\,h \times D/h \\ &= a_p/D\,.\,h \times \delta\,; \quad \text{.............. 10.2} \end{aligned}$$

and $$\begin{aligned} a_p/D\,.\,h &= a_p\,.\,a_{p-1}/D\,.\,h + a_p\,.\,\bar{a}_{p-1}/D\,.\,h \\ &= a_p/a_{p-1}\,.\,D\,.\,h \times a_{p-1}/D\,.\,h \\ &\qquad + a_p/\bar{a}_{p-1}\,.\,D\,.\,h \times \bar{a}_{p-1}/D\,.\,h \\ &= \alpha_p \times a_{p-1}/D\,.\,h + (1-\alpha_p) \times \bar{a}_{p-1}/D\,.\,h, \quad \text{.......... 10.3} \end{aligned}$$

assuming, as we did before, that the actual truth or falsehood of D's statement is irrelevant to the truth of the secondary witnesses' reports.

Similarly

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Whence immediately

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Hence

$$a_p/h - \bar{a}_p/h = (2\alpha_p - 1) \dots (2\alpha_1 - 1)(2\delta - 1),$$

and

$$\left.\begin{aligned} a_p/h &= \tfrac{1}{2}\{1 + (2\alpha_p - 1) \dots (2\alpha_1 - 1)(2\delta - 1)\} \\ \bar{a}_p/h &= \tfrac{1}{2}\{1 - (2\alpha_p - 1) \dots (2\alpha_1 - 1)(2\delta - 1)\} \end{aligned}\right\}. \quad \dots\dots\dots 10.7$$

From 10.1, 10.2, 10.6 and 10.7 we have

$$\left.\begin{aligned} \frac{D}{a_p \,.\, h} &= \frac{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \dots (2\alpha_1 - 1)}{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \dots (2\alpha_1 - 1)(2\delta - 1)}\,\delta \\ \text{and} \quad & \\ \frac{D}{\bar{a}_p \,.\, h} &= \frac{1 - (2\alpha_p - 1)(2\alpha_{p-1} - 1) \dots (2\alpha_1 - 1)}{1 + (2\alpha_p - 1)(2\alpha_{p-1} - 1) \dots (2\alpha_1 - 1)(2\delta - 1)}\,\delta \end{aligned}\right\}. \quad \dots\dots 10.8$$

Two points may be noted.

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"C says that D speaks truth" will be represented by a round block, and "C says that D lies" by a square block; each of these blocks has a hole in it, into which one of the D blocks will fit; a true statement will be represented by a block of the same shape as the hole in it, and a falsehood by a block of a different shape from

its hole. Similarly A's and B's statements will be represented by blocks fitting over the B and C blocks respectively, a block of the same shape as that over which it fits representing a true statement and one of a different shape a false statement. Then a set of blocks fitted together and terminating in a round A block will represent the statement in the problem, and one terminating in a square A block the alternative (it has already been shown that there are only two alternatives).

The model will also illustrate the distinction already explained between the two definitions of a witness's veracity or credibility. Consider the witness A, and suppose the blocks picked out of a store containing R_r, R_s, S_r and S_s blocks of the forms round outside round inside, round outside square inside, square outside round inside, and square outside square inside respectively. Then if we define A's veracity as the probability that if we pick a block at random to fit outside a B block, this block will be the same shape inside and out, this probability will be $R_r/(R_r+S_r)$ if the B' block is round and $S_s/(R_s+S_s)$ if the B block is square. Our assumption that the veracity is independent of the statement made is equivalent to assuming that these two fractions are equal, and no further difficulty arises. The other definition is represented by the probability that a block the outside shape of which is given is of the same inside shape and is $\dfrac{R_r}{R_r+R_s}$ or $\dfrac{S_s}{S_r+S_s}$, which is clearly generally different from the former, though it can be made the same by putting $S_r=R_s$, which, with the former, implies $R_r=S_s$. We find no difficulty in proceeding down the series in this way until we come to the D block, but here we stop, because the inside of the C block fixes the nature of the D block and D's veracity has no scope. It might be thought that this could be cured by giving D four sorts of blocks, as the other witnesses have; but this is contrary to the conditions of the problem, according to which the other witnesses have each four types of statement open to them, two true and two false, whereas D has only two. H. W. C.

GLEANINGS FAR AND NEAR.

1076. Scotland's New-style Goalkeeper.

Magnificently built, and with every physical advantage in his favour, Brown does most of his work many yards out of goal. His method may look daring at times, but there is common sense, and even sound mathematics, behind it.—*Sunday Referee*, February 9, 1936. [Per Mr. C. A. Richmond.]

1077. Picture of a Mathematician !

The eyes, slightly Mongoloid in cast, had the contemplative vision which penetrated through an object to abstract principles behind. The owner of such eyes, thought Tommy, could readily enough be associated with the fine-spun mathematical sequences contained in this treatise.—Alice Campbell, *Desire to Kill*, p. 15. [Per Mr. J. B. Bretherton.]

THE A, B, C, D PROBLEM.

BY H. S. LEFTWICH.

THE problem is: "A affirmed that B denied that C declared that D was a liar. If A, B, C and D each speak the truth (independently) once in three times, what is the probability that D was speaking the truth?"

This problem, first propounded, I believe, by Sir A. Eddington, has aroused considerable interest, and a solution of it was given by Sir A. Eddington in the *Gazette*, October 1935. His method of attack was, however, somewhat unusual, and the object of this contribution is to show how the problem may be attacked on normal lines, and a perfectly general solution obtained. For this purpose let us suppose that the chance that A speaks the truth is a, that B speaks the truth is b, and similarly with C and D.

Now we know that A affirmed that B denied, etc.

(1) Therefore chance that B denied and "truthed" is ab.

(2) And chance that B denied and lied is $a(1-b)$.

We cannot say whether B affirmed, etc. There is no information given as to this, and it is quite possible that B said nothing at all. If he did not deny, we are not entitled to assume that he therefore affirmed.

Thus in case (1) C did not declare, etc., and we are not further concerned with this case; and in case (2) C did declare, etc.

(3) Chance that C declared and truthed $= ac(1-b)$.

(4) Chance that C declared and lied $= a(1-b)(1-c)$.

The sum of these two chances is $a(1-b)$.

(5) Thus chance that C neither declared nor denied (let us say that he was silent) is $1-a(1-b)$.

We are now able to make the following statements about C and D:

(6) Chance that C truthed and D truthed $= ac(1-b)d$.

(7) Chance that C truthed and D lied $= ac(1-b)(1-d)$.

(8) Chance that C lied and D truthed $= a(1-b)(1-c)d$.

(9) Chance that C lied and D lied $= a(1-b)(1-c)(1-d)$.

(10) Chance that C was silent and D truthed $= \{1-a(1-b)\}d$.

(11) Chance that C was silent and D lied $= \{1-a(1-b)\}(1-d)$.

Case (6) contradicts the enunciation of the problem, and so does case (9). The remaining possible chances, cases (7), (8), (10) and (11), add up to $1-a(1-b)+a(1-b)(c-2cd+d)=p$ (say). Since we know that D said something, this chance becomes certainty, and in cases (8) and (10) D told the truth. The sum of cases (8) and (10) is

$$d-acd(1-b).$$

But by the principle of inverse probability, this chance must be multiplied by $\frac{1}{p}$.

Hence final chance that D told the truth is

$$\frac{d-acd(1-b)}{1-a(1-b)+a(1-b)(c-2cd+d)}=P \text{ (say)}.$$

If in this formula we put $a=b=c=d=\frac{1}{3}$, it works out to $\frac{25}{71}$, the value given by Eddington.

The formula may be checked in various ways. Thus :

If

$$d=0, \text{ then } P=0;$$

$$d=1,\ P=\frac{1-ac(1-b)}{1-ac(1-b)}=1.$$

If $a=0$, the original statement becomes " B did not deny that C declared that D was a liar ", in which case the chance that D spoke the truth is simply d. The formula bears this out.

If $b=1$, we know that C did not declare that D was a liar. Again the probability is d, and the formula shows it.

If $c=\frac{1}{2}$, this is equivalent to saying that C is just as likely to tell the truth as to tell a lie, and hence his remarks do not matter. Consequently those of A and B do not matter either, and the chance that D spoke the truth is d. Again the formula bears this out.

H. S. Leftwich.

1078. Any engineer also knows that the energy of a moving body is the product of its mass by its velocity. If you have a car weighing one ton travelling at 60 miles an hour its energy is 197,120 foot-pounds per second. That is a great deal of energy.

But if the car is only travelling at 30 miles an hour its energy will be 98,560 foot-pounds per second. Only half as much, you see. Which means that the car travelling 30 miles an hour will only cause half as much damage if it hits anything as the car travelling 60 miles an hour.

Because, whether you believe it or not, it's the kinetic energy that does the damage. You may think it's the bumpers, or the wings, or the wet road that made you skid, or the other half of the last one. It isn't any of those things. It's the kinetic energy.—*News-Chronicle*, October 11, 1935. [Per Messrs. T. Marsden and A. H. G. Palmer.]

1079. A motor travelling at 20 m.p.h. hits a pedestrian with four times the blow that it does when travelling at 10 m.p.h. Double the speed and you quadruple the force of the blow.—From a broadcast by Mr. L. Hore-Belisha. [Per Mr. H. V. Lowry.]

1080. At the height of his [D'Alembert's] fame she [his adopted mother] remonstrated with him for wasting his talents on such work : " Vous ne serez jamais qu'un philosophe ", said she, " et qu'est-ce qu'un philosophe ? C'est un fou qui se tourmente pendant sa vie, pour qu'on parle de lui lorsqu'il ne sera plus ".—Rouse Ball, *History of Mathematics*, p. 375. [Per Dr. G. J. Lidstone.]

A PROBLEM ON RANDOM PATHS.

By W. H. McCrea.

1. *The general problem.* This paper deals with particular cases of the following problem: A rectangular array of square cells is given; a particle P moves from one cell to another in such a way that, when it is in any cell, it is equally likely to pass across any of its edges. P is liberated in any given cell, and *it is required to find the probability that it will ultimately emerge across any stated edge in the boundary of the array.*

The problem presented itself as a very drastic schematization of the wanderings of a quantum of radiation in a scattering medium or fog, but its possible significance in that connection is not discussed here.

It is evident that P must in fact ultimately emerge across some outside edge, in the sense that if it makes, say, one jump per unit time then the chance that it remains inside the array after time t tends to zero as $t \to \infty$. For wherever P is in the array there is certainly one finite route by which it could escape to the outside and a finite probability that it will actually take this route. (If the route consists of ν jumps this probability is $(1/4)^{\nu}$.) So at any epoch a finite time interval can be specified after which the chance that P is still in the cells is less than unity by a finite amount. Therefore the resultant probability that it is still inside after a sufficiently large number of such intervals can be made arbitrarily small.

If P is restricted to move only, say, " forward " or " to the right ", then the number of routes available becomes finite instead of infinite, and the problem reduces to a well-known one.* A problem on random paths in which the number of possibilities is unlimited as here is Karl Pearson's problem of the " random walk " solved by Rayleigh.† However, that problem has nothing to do with the crossing of a boundary, and its solution is of no assistance in the present case, which does not appear to have been treated before.

It does not seem to be practicable to give an exact solution for a general number of cells. But it is possible to do so for any number, finite or infinite, in a single or a double row, and this is done in the present paper. Approximate solutions could be got for other cases, but these will not be treated.

We have three simple general properties which hold for any form of array which may first be noticed:

(i) *If p_i is the probability that P emerges across an outer edge i, then the sum of p_i for all i is unity.* This merely restates the result that P must ultimately emerge somewhere.

(ii) *A single cell may possess more than one " outside " edge; if so*

* W. W. Rouse Ball, *Mathematical Recreations and Essays* (9th ed., 1920), 135; E. Borel, *Calcul des Probabilités*, Tome 1, Fasc. 1 (1925), Ch. V.

† Rayleigh, *Scientific Papers*, V, 256.

the quantities p_i *are equal for all these edges.* This is trivial, for P has got to get into the cell in question before it can emerge across its outside edges, and once it is in the cell it is equally likely to cross any of these edges. Thus, in Fig. 1, p_i is the same for all the edges marked 1, for both marked 2, and so on, so that for this array the result (i) gives $3p_1+2p_2+2p_3+2p_4+p_5+2p_6+p_7+p_8=1$.

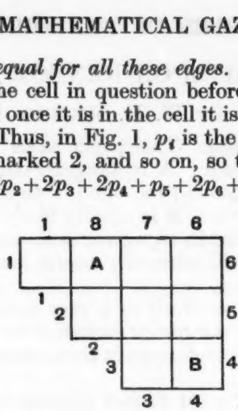


FIG. 1.

(iii) *If A, B be any two cells of the system, then the probability that P, starting from A, ultimately reaches B, is equal to the probability that, starting from B, it ultimately reaches A.* For consider any possible route joining A, B, consisting in all of, say, ν jumps, including any part of the path which may be retraced any number of times. If P starts from A then the chance that it takes the first of these jumps is $\frac{1}{4}$, and if it does so the chance that it then takes the next is again $\frac{1}{4}$, and so on. Hence the chance that it takes the whole route is $(\frac{1}{4})^\nu$. Similarly, if P starts from B the chance that it follows this route is also $(\frac{1}{4})^\nu$. But the totality of routes leading from A to B is the same as that of routes from B to A. Hence the result stated. In particular, it follows from this that, *if A, B both possess "outside" edges, the probability that P, starting from A, ultimately emerges across any particular outside edge of B, is the same as the probability that, starting from B, it ultimately emerges across any particular outside edge of A.*

Single row of cells.

2. *Starting in end cell.*

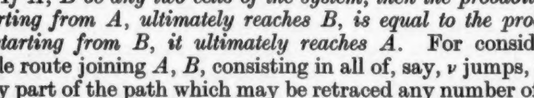


FIG. 2.

Suppose we have n cells in a row, with edges numbered as in Fig. 2. Let P start in the left-hand cell (X) and let $p_i^{(n)}$ be the probability that it ultimately leaves the system across an edge i. Then the chance that P crosses an edge 1 in the first jump is $\frac{3}{4}$. Also the chance that it goes into the second cell in the first jump is $\frac{1}{4}$, and if it does, the chance that it eventually returns to the first cell is $p_1^{(n-1)}$, by definition of the latter quantity. If it does return,

the chance that it finally crosses an edge 1 is again $p_1^{(n-1)}$. Combining these results we find

$$p_1^{(n)} = \tfrac{1}{4} + \tfrac{1}{4} p_1^{(n-1)} p_1^{(n)}. \quad \text{.........................(1)}$$

Again, if P enters the second cell, the chance that it then crosses an edge i without returning to its starting point is $p_{i-1}^{(n-1)}$, while the chance that it does return is $p_1^{(n-1)}$. If it does, the chance that it finally crosses an edge i is again $p_i^{(n)}$. Putting together these possibilities we have

$$p_i^{(n)} = \tfrac{1}{4} p_{i-1}^{(n-1)} + \tfrac{1}{4} p_1^{(n-1)} p_i^{(n)}, \quad (i > 1). \quad \text{...............(2)}$$

These give non-linear difference equations for $p_i^{(n)}$. Equation (1) may be turned into a linear equation by looking for a solution of the form $p_1^{(n)} = f_n/f_{n+1}$, which is seen to exist if

$$4f_n = f_{n-1} + f_{n+1}. \quad \text{..............................(3)}$$

We know $p_1^{(1)} = \tfrac{1}{4}$, and hence can get $p_1^{(2)} = 4/15$ directly from (1), whence, using these to fix the constants in the general solution of (3), the required solution is

$$f_n = [(2+\sqrt{3})^n - (2-\sqrt{3})^n]/2\sqrt{3}, \quad \text{...................(4)}$$

giving the sequence 1, 4, 15, 56, ... This suggests looking for a solution of (2) of the form $g_i^{(n)}/f_{n+1}$, and we find in fact that it is satisfied if $g_i^{(n)} = g_{i-1}^{(n-1)} = \ldots = g_1^{(n-i+1)} \equiv f_{n-i+1}$, using the condition that $g_1^{(n-i+1)}/f_{n-i} = p_1^{(n-i+1)} = f_{n-i+1}/f_{n-i}$. So finally

$$p_i^{(n)} = \frac{f_{n-i+1}}{f_{n+1}} = \frac{(2+\sqrt{3})^{n-i+1} - (2-\sqrt{3})^{n-i+1}}{(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}}. \quad \text{............(5)}$$

It is readily verified that this satisfies the property (i). As examples, Fig. 3 shows the probabilities for each edge for $n = 1, \ldots, 4$.

The rule for the *relative* probabilities is merely to write 1, 4, 15, ..., starting from the end opposite the starting point of P.

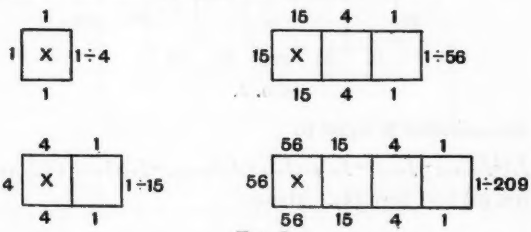


FIG. 3.

Semi-infinite row. If in (5) we let $n \to \infty$, we get

$$p_i^{(n)} \to p_i^{(\infty)} = (2+\sqrt{3})^{-i} = (2-\sqrt{3})^i, \quad \text{................(6)}$$

giving a simple result of rather unexpected form, which can easily be obtained by writing down equations corresponding to (1), (2) for this case.

3. *General starting point.*

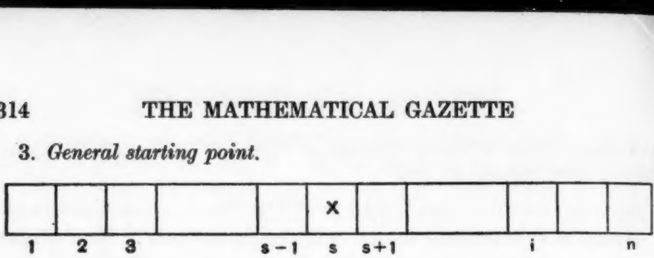


FIG. 4.

Now consider the case of n cells in a row, when P starts in any cell s (Fig. 4). Let ${}_sp_i^{(n)}$ be the probability that it eventually escapes across an edge i.

($i > s$.) The chance that P goes first into cell $(s-1)$ is $\frac{1}{4}$, and if so the chance that it eventually returns to its starting point is ${}_1p_1^{(s-1)}$. Also the chance that P goes first into cell $(s+1)$ is $\frac{1}{4}$, and if so the chance that it crosses an edge i before returning to its starting point is ${}_1p_{(i-s)}^{(n-s)}$, while the chance that it does return is ${}_1p_1^{(n-s)}$. If it returns the chance that it ultimately crosses edge i is again ${}_sp_i^{(n)}$. Combining all these we find

$$ {}_sp_i^{(n)} = \tfrac{1}{4}\,{}_1p_{(i-s)}^{(n-s)} + \tfrac{1}{4}\left({}_1p_1^{(s-1)} + {}_1p_1^{(n-s)}\right){}_sp_i^{(n)}. \qquad \text{............(7)} $$

The quantities ${}_1p_i^{(n)}$ are what we have called $p_i^{(n)}$ in § 2. So (7) gives ${}_sp_i^{(n)}$ immediately in the form, using (5),

$$ {}_sp_i^{(n)} = \frac{f_s f_{n-i+1}}{4f_{n-s+1}f_s - f_{n-s}f_s - f_{n-s+1}f_{s-1}}. $$

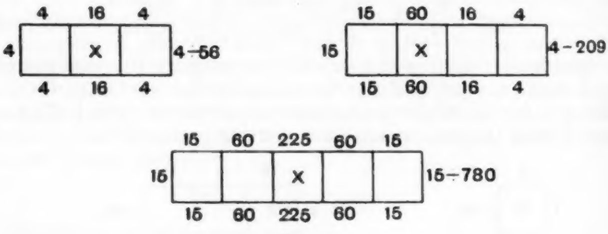


FIG. 5.

The denominator is equal to

$$ f_s(4f_{n-s+1} - f_{n-s}) - f_{n-s+1}f_{s-1} = f_s f_{n-s+2} - f_{s-1}f_{n-s+1} = f_{n+1}, $$

using first (3) and then (4). Hence

$$ {}_sp_i^{(n)} = \frac{f_s f_{n-i+1}}{f_{n+1}}, \quad (i > s), \qquad \text{......................(8·1)} $$

$$ {}_sp_i^{(n)} = \frac{f_i f_{n-s+1}}{f_{n+1}}, \quad (i < s), \qquad \text{......................(8·2)} $$

and

$$ {}_sp_s^{(n)} = \frac{f_s f_{n-s+1}}{f_{n+1}}. \qquad \text{...............................(8·3)} $$

(8.2) follows because ($i<s$) can be turned into ($i>s$) by numbering from the other end. Derived in this way it verifies the truth of (iii) in this particular case. (8.3) is obtained by putting $i=s$ in (8.1), (8.2), but actually it requires to be proved directly, which is easy. Examples of these formulae are shown in Fig. 5, giving the probabilities associated with each edge when P starts from X. Fortuitous factors are not cancelled, to facilitate comparison with Fig. 3.

Semi-infinite row. Letting $n \to \infty$ in (8), we find

$$ {}_sp_i^{(n)} \to {}_sp_i^{(\infty)} = (2-\surd 3)^i f_s, \quad (i \geqslant s), \qquad (9{\cdot}1)$$

$$ {}_sp_i^{(n)} \to {}_sp_i^{(\infty)} = (2-\surd 3)^s f_i, \quad (i \leqslant s). \qquad (9{\cdot}2)$$

Infinite row. We can get the case of a row of cells extending to infinity in both directions by letting n, i, $s \to \infty$ in (8), keeping $j=|i-s|$ finite. Then, since from (4)

$$f_n \sim (2+\surd 3)^n/2\surd 3, \quad (n \to \infty),$$

both (8·1), (8·2) give

$$ {}_sp_i^{(n)} \to p_j = (2-\surd 3)^j/2\surd 3, \qquad (10{\cdot}1)$$

and (8.3) gives

$$ {}_sp_s^{(n)} \to p_0 = 1/2\surd 3. \qquad (10{\cdot}2)$$

Here p_j is the probability that P, starting from cell 0 will eventually emerge across an edge j (Fig. 6). The results (9), (10) are not difficult to prove directly.

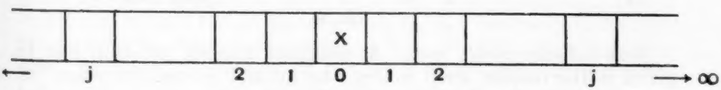


FIG. 6.

4. *Double row of cells.*

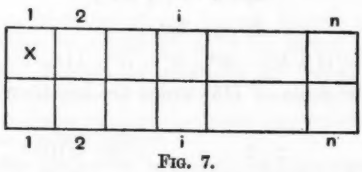


FIG. 7.

Consider now a double row of $2 \times n$ cells, and suppose P starts from the top left-hand corner (Fig. 7). Let $q_i^{(n)}$, $r_i^{(n)}$ be the chance that it emerges eventually across an edge i in the top or bottom row.

The chance that P goes first into the second cell in the top row is $\frac{1}{4}$, and if it does so the chances that it leaves the $2 \times (n-1)$ block across edges i are $q_{i-1}^{(n-1)}$, $r_{i-1}^{(n-1)}$, while the chances that it returns to the first top and bottom cells are $q_1^{(n-1)}$, $r_1^{(n-1)}$. If it returns to the top one, the chances that it finally emerges across top or bottom i

are again $q_i^{(n)}$, $r_i^{(n)}$, and if it returns to the bottom one they are $r_i^{(n)}$, $q_i^{(n)}$.

The chances that P goes into the first bottom cell in the first jump is also $\frac{1}{4}$, and if it does so the chances that it finally crosses top or bottom edge i are again $r_i^{(n)}$, $q_i^{(n)}$. Putting all these possibilities together we have ($i>1$),

$$q_i^{(n)} = \tfrac{1}{4}q_{i-1}^{(n-1)} + \tfrac{1}{4}r_i^{(n)} + \tfrac{1}{4}q_1^{(n-1)}q_i^{(n)} + \tfrac{1}{4}r_1^{(n-1)}r_i^{(n)}, \quad \text{.....(11·1)}$$

$$r_i^{(n)} = \tfrac{1}{4}q_i^{(n)} + \tfrac{1}{4}r_{i-1}^{(n-1)} + \tfrac{1}{4}q_1^{(n-1)}r_i^{(n)} + \tfrac{1}{4}r_1^{(n-1)}q_i^{(n)}, \quad \text{.....(11·2)}$$

or

$$q_i^{(n)}(4 - q_1^{(n-1)}) - r_i^{(n)}(1 + r_1^{(n-1)}) = q_{i-1}^{(n-1)}, \quad \text{..........(12·1)}$$

$$-q_i^{(n)}(1 + r_1^{(n-1)}) + r_i^{(n)}(4 - q_1^{(n-1)}) = r_{i-1}^{(n-1)}. \quad \text{..........(12·2)}$$

These hold for all i if we make the convention $q_0^{(n-1)} = 1$, $r_0^{(n-1)} = 0$. I have not obtained a general solution of (12) as functions of n, i. But they may be solved in succession for $n = 1, 2, 3, \dots$; the results up to $n = 3$ are given in Fig. 8.

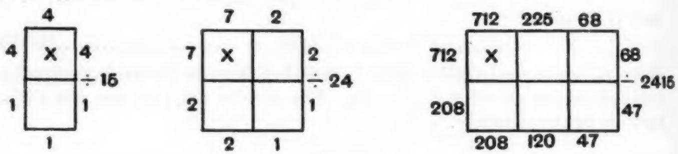


FIG. 8.

Semi-infinite double row. A complete explicit solution can be given if the double strip extends to infinity in one direction, *i.e.* $n \to \infty$ in Fig. 7. For let $q_1^{(n)} \to q$, $r_1^{(n)} \to r$, when $n \to \infty$, then equations (11) become ($i = 1$)

$$4q = 1 + r + q^2 + r^2,$$

$$4r = q + 2qr,$$

or $$q = 4r/(1+2r), \quad 4r^4 + 8r^3 - 7r^2 - 11r + 1 = 0. \quad \text{............(13)}$$

We require the roots of (13) which are less than 1, and these are given by

$$q = 2 - \frac{2\sqrt{2}}{\sqrt{13 - \sqrt{105}}}, \quad r = \frac{\sqrt{13 - \sqrt{105}} - \sqrt{2}}{2\sqrt{2}}. \quad \text{..........(14)}$$

These are given exactly, instead of giving decimals, as a matter of curiosity; it is amusing to find that a simple question of probability should lead to this kind of number.

With $i > 1$, equations (12) become

$$q_i(4 - q) - r_i(1 + r) = q_{i-1}, \quad \text{..................(15·1)}$$

$$-q_i(1 + r) + r_i(4 - q) = r_{i-1}, \quad \text{..................(15·2)}$$

where $q_i^{(n)} \to q_i$, $r_i^{(n)} \to r_i$, when $n \to \infty$. There are a pair of simul-

taneous linear difference equations in q_i, r_i. Solving them as such we find ultimately

$$q_i = \tfrac{1}{2}(\alpha^i + \beta^i), \quad r_i = \tfrac{1}{2}(\alpha^i - \beta^i), \quad \text{.................(16)}$$

(i on the right-hand side being an exponent and not a suffix) where

$$\alpha = 1/(3-q-r), \quad \beta = 1/(5-q+r), \quad \text{...............(17)}$$

q, r being given by (14).

Infinite double row. This case can also be solved using the results just given. Let P start from X in Fig. 9, and let q_0, r_0 be the

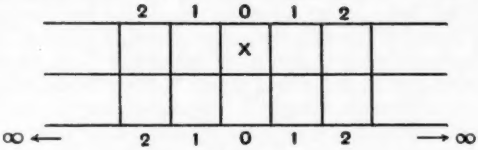


Fig. 9.

probabilities that it ultimately leaves the strip across edge 0 in the top or bottom rows. Then the chance that P jumps first across edge 0 in the top row is $\frac{1}{4}$. Also the chance that it jumps first to the right is $\frac{1}{4}$. If it does so the chance that it ultimately returns to its starting point is given by the quantity q already found, and if it does return the chances that it eventually crosses top or bottom 0 are again q_0, r_0; the chance that it ultimately returns to the cell below its starting point is given by r, and then the chances that it eventually crosses top or bottom 0 are r_0, q_0. Similarly if it jumps first to the left. Or it may jump first into the cell below its starting point, again giving chances r_0, q_0 that it ultimately emerges across top or bottom 0. These results are expressed by the equations

$$q_0 = \tfrac{1}{4} + \tfrac{1}{4}r_0 + \tfrac{1}{2}qq_0 + \tfrac{1}{2}rr_0, \quad \text{......................(18·1)}$$

$$r_0 = \tfrac{1}{4}q_0 + \tfrac{1}{2}qr_0 + \tfrac{1}{2}rq_0 \quad \text{...........................(18·2)}$$

These give q_0, r_0 in terms of the known quantities q, r, and the solution could then be completed as in the preceding case.

5. *Other cases.* Recurrence relations analogous to (11) can be written down for the general rectangular array, but it scarcely seems possible to solve them exactly for cases appreciably more complicated than those already given.

The problem can, however, be stated for any network of polygons, by saying that when the particle P is in an N-gon there is a probability $1/N$ that it will jump across any one of its edges. The general case would present immense difficulty. But results like those of the present paper can just as well be given for, say, rows of *triangles*. The simplest case of all is that of 2-gons, or *parallel grooves*. If P is liberated in the sth groove of a set of n, then the chances that it eventually emerges across the outside edge of the first or the last are

$$(n-s+1)/(n+1) \quad \text{and} \quad s/(n+1).$$

W. H. McC.

SOME DIFFICULT SARACENIC DESIGNS. III.

A PATTERN CONTAINING FIFTEEN-RAYED STARS

By E. Hanbury Hankin

The design shown in Fig. 1 is unusual in the amount of pattern that

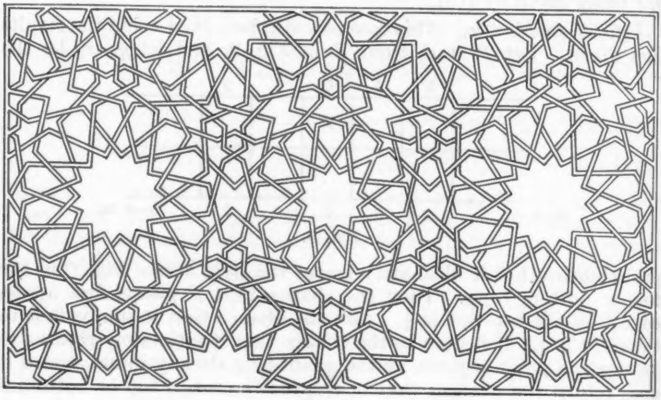


Fig. 1.

goes to one repeat, only one complete repeat being included in the illustration. It is also unusual in that it includes fifteen-rayed stars.

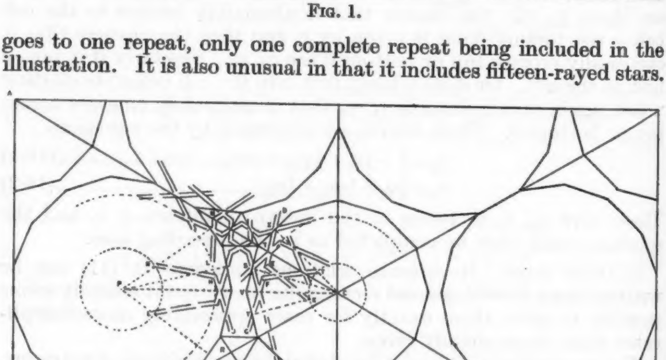


Fig. 2.

As indicated in Fig. 2, the repeat occupies a rectangle whose diagonal makes angles of 30 and 60 degrees with the adjacent sides. One begins the construction lines by drawing on such a rectangle a

lattice of equilateral triangles. One of these triangles is indicated by the letters *ABC*. In each of these triangles draw a fifteen-sided polygon. Dodecagons are then described whose centres correspond to the angles of each equilateral triangle. These dodecagons are of such a size that the length of their sides is equal to the length of the side of a 15-gon. Adjacent corners of the 15-gons and 12-gons are joined by lines forming pairs of small triangles as shown at *R*. The main construction lines are thereby completed. The pattern is constructed by drawing two lines crossing each other through the centres of each side of each polygon and triangle.

It is advisable to begin with the lines *DE* and *FG*. These are in line with each other and each passes through the centre of one of the sides of the 12-gon. Inside the 12-gon these lines come to an end at the points *E* and *F*, which are at a small distance, that must be guessed, from the interradius *SC*. From the centre *C* at the distance *E* or *F* describe a circle; this is partly shown drawn as a dotted line. All the other lines that enter the 12-gon from the outside terminate on points on this circle at the same distances from interradii as the two lines already drawn. From the point *E* draw the line *EK* nearly parallel with the radius *CT*. This line comes to an end where it meets the interradius *CY*. From the centre *C* draw a circle intersecting the point *K*. All the other lines that come towards the centre of the 12-gon end on this circle at points where it cuts interradii. Similarly the line *FT* is drawn nearly parallel to the radius *CV*. Other lines drawn in the same way complete the twelve-rayed star of the pattern that occupies the 12-gon construction outline.

Now let us go to the fifteen-rayed star that is to occupy the 15-gon construction outline. Begin with drawing the pattern line *HJ*. This is not drawn in line with the line *ML* but, passing through the centre of a side of the 15-gon, it is given an inclination that makes it as much as possible a looking-glass reflection of the line *FG*. The line *HJ* comes to an end inside the 15-gon at a point near an interradius. From the centre *P* describe a circle, which is shown partly drawn as a dotted line, passing through the point *H*. All other lines that enter the 15-gon end on this circle at points similarly near interradii. From the point *H* a line is drawn nearly parallel to the neighbouring radius and ending on the next interradius at *N*. From the centre *P* at distance *N* draw a circle. All other lines that approach the centre of the 15-gon end on this circle at points where it cuts interradii.

The remaining pattern lines are those that form small hexagons that occupy each of the triangles of the construction lines. These lines cross the centres of the sides of the triangles and are so drawn as to make the resulting hexagons as nearly symmetrical as possible.

This pattern may be found in *Les Eléments de l'Art Arabe : Le trait des Entrelacs*, by J. Bourgoin (Paris, Librairie de Firmin-Didot et Cie, 1879), Plate 128. It is also briefly described in my paper, " The Drawing of Geometric Patterns in Saracenic Art ", in *Memoirs of the Archaeological Survey of India*, No. 15 of 1925, Fig. 39.

E. H. H.

ON CERTAIN RELATED CURVES.

BY C. E. WEATHERBURN.

THE relation between a curve and its involute suggests the idea of a pair of curves such that *the binormals of one cut the other orthogonally.* Let $\mathbf{r}$ be the position vector of the current point P on the first curve C and $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the unit vectors in the directions of the tangent, the principal normal and the binormal at P. These are functions of the arc-length s of the curve. Since the binormals of C intersect the second curve C_1, the position vector of the point P_1 of the second curve which lies on the binormal at P is given by

$$\mathbf{r}_1 = \mathbf{r} + a\mathbf{b}, \quad \text{.....(1)}$$

in which a is a scalar. Let the suffix unity be used to distinguish quantities relating to C_1. Then the unit tangent to this curve at P is

$$\mathbf{t}_1 = \frac{d\mathbf{r}_1}{ds}\frac{ds}{ds_1} = (\mathbf{t} + a'\mathbf{b} - a\tau\mathbf{n})\frac{ds}{ds_1},$$

τ being the torsion of C at P. In order that the condition of orthogonality may be satisfied, this vector must be perpendicular to $\mathbf{b}$, so that $a' = 0$ and therefore a is constant. Since $\mathbf{t}_1$ is a unit vector, it follows from the last equation that

$$\mathbf{t}_1 = \frac{\mathbf{t} - a\tau\mathbf{n}}{\sqrt{(1 + a^2\tau^2)}} \quad \text{.....(2)}$$

and
$$\frac{ds_1}{ds} = \sqrt{(1 + a^2\tau^2)}. \quad \text{.....(3)}$$

Thus for a given curve C, because the constant a is arbitrary, there is a single infinitude of curves C_1 having the specified property.

A particular case of some interest is that in which *the binormals of C are principal normals of C_1.* In order that this relation may hold, the curvature and torsion of C must satisfy a certain differential equation, as appears from the following. If (2) be differentiated with respect to s_1 we obtain an expression for $\kappa_1\mathbf{n}_1$ as the sum of components in the directions of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. In order that $\mathbf{n}_1$ may be parallel to $\mathbf{b}$, the coefficients of $\mathbf{t}$ and $\mathbf{n}$ must vanish. This will be the case if the relation

$$\kappa(1 + a^2\tau^2) = a\frac{d\tau}{ds} \quad \text{.....(4)}$$

is satisfied by the curvature and torsion of C. We may write this

$$\frac{d}{ds}\arctan a\tau = \kappa$$

or
$$a\tau = \tan\left(\int\kappa\, ds + c\right),$$

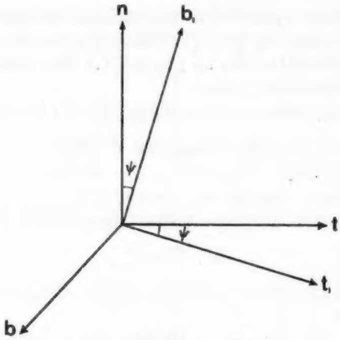
and give the condition a geometrical interpretation as follows. Consider the developable surface generated by the tangents to C,

i.e. the osculating developable of C. When this is unwrapped into a plane let ψ be the inclination of the tangent at the current point P to *any* fixed tangent. Then $\kappa = d\psi/ds$, and the above condition is equivalent to

$$a\tau = \tan\psi, \qquad\qquad (5)$$

and (3) shows that

$$ds_1/ds = \sec\psi.$$



When the condition (4) is satisfied we obtain, on differentiating (2) with respect to s_1,

$$\kappa_1 \mathbf{n}_1 = -\frac{a\tau^2 \mathbf{b}}{1 + a^2\tau^2}.$$

We may choose

$$\mathbf{n}_1 = -b_1, \qquad\qquad (6)$$

so that the curvature of C_1 has the value

$$\kappa_1 = \frac{a\tau^2}{1 + a^2\tau^2} = \frac{1}{a}\sin^2\psi. \qquad\qquad (7)$$

The unit binormal to C_1 is then

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \frac{\mathbf{n} + a\tau\mathbf{t}}{\sqrt{(1 + a^2\tau^2)}}. \qquad\qquad (8)$$

The torsion of C_1 is found by differentiating (8) with respect to s_1. We thus obtain

$$-\tau_1 \mathbf{n}_1 = \tau\mathbf{b}/(1 + a^2\tau^2),$$

the coefficients of $\mathbf{t}$ and $\mathbf{n}$ vanishing in virtue of (4). Consequently

$$\tau_1 = \frac{-\tau}{1 + a^2\tau^2} = -\frac{1}{a}\sin\psi\cos\psi. \qquad\qquad (9)$$

From (2), (5) and (8) it is clear that the inclination of $\mathbf{t}$ to $\mathbf{t}_1$ and of $\mathbf{n}$ to $\mathbf{b}_1$ is the angle ψ found above. Also if the tangent surface to C is developed into the osculating plane at P, the direction of $\mathbf{t}_1$ is that of the line from which ψ is measured.

C. E. Weatherburn.

A GENERALIZATION OF PTOLEMY'S THEOREM.

By L. M. Milne-Thomson.

The three-dimensional analogue of Ptolemy's theorem (Simson's Euclid, Prop. D) has been given by Cayley (*Collected Works*, I, 1). The following note contains the theory of such generalizations to space of any number of dimensions.

Taking Euclidean space of n dimensions we denote the Cartesian coordinates of a point A_s by $(x_{s,\alpha})$ where the Greek suffix is a dummy and takes the values $1, 2, 3, \dots, n, n+1$. We make the convention that $x_{s, n+1}=1$ for every point A_s.

The distance a_{st} between the points A_s, A_t is defined by

$$a_{st}^2=(x_{s,\alpha}-x_{t,\alpha})^2=a_{ts}^2$$

and therefore $a_{ss}=0$.

If we take $n+1$ points A_s $(s=1, 2, \dots, n+1)$ and choose as origin that point which is equidistant from them all, we have

$$1+R^2=x_{s,\alpha}^2,$$

where R denotes the common value of the distance.

It follows that

$$x_{s,\alpha}x_{t,\alpha}=1+R^2-\tfrac{1}{2}a_{st}^2.$$

Consider the determinant

$$W_n=|\,x_{\alpha,\beta}\,|$$

where α denotes the row, β the column.

Squaring this determinant by multiplying row by row, we have

$$\begin{aligned} W_n^2&=|\,x_{\alpha,\gamma}x_{\beta,\gamma}\,|\\ &=|\,1+R^2-\tfrac{1}{2}a_{\alpha\beta}^2\,|, \end{aligned}$$

and therefore

$$2^{n+1}W_n^2=|\,2R^2+2-a_{\alpha\beta}^2\,|.$$

If we border this determinant by adjoining a zero to the right of every row, a term $2R^2+2$ at the foot of every column but unity at the lower end of the leading diagonal, we have, by subtracting the new last row from every row,

$$2^{n+1}W_n^2=(2R^2+2)\,|\,a_{\alpha\beta}^2\,|_B+|\,a_{\alpha\beta}^2\,|\,(-1)^{n+1},$$

where B suffix denotes that the determinant is to be bordered with units on the top and on the right and zero at the upper end of the ascending diagonal.

The bordered determinant is homogeneous and of degree $2n$ in the a_{st} while the unbordered determinant is of degree $2n+2$ and is likewise homogeneous. Hence we have

(1) $$2^nW_n^2=|\,a_{\alpha\beta}^2\,|_B.$$

(2) $$R^2=-(-2)^{-n-1}\,|\,a_{\alpha\beta}^2\,|\div W_n^2.$$

If we regard W_n as a measure of the hypervolume of the n dimensional simplex and R as the radius of the circumscribing hypersphere, the volume and radius are determined by (1) and (2) in terms of the lengths of the joins of the pairs of vertices.

If $W_n = 0$, the $n+1$ vertices of the simplex lie in a space of $n-1$ dimensions, and

$$|a_{\alpha\beta}^2|_B = 0 \tag{3}$$

then gives the identical relation which must subsist between the mutual distances of $n+1$ arbitrary points in a space of $n-1$ dimensions. In this case R becomes infinite. If, however, we have in addition

$$|a_{\alpha\beta}^2| = 0, \tag{4}$$

R is indeterminate and the $n+1$ points then lie on a hypersphere in space of $n-1$ dimensions. Equation (4) then constitutes a generalization of Ptolemy's theorem on the cyclic quadrangle.

Taking $n=3$, the simplex is a tetrahedron whose volume $V = \frac{1}{6} W_3$ is given by

$$288V^2 = \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 & 1 \\ a_{21}^2 & 0 & a_{23}^2 & a_{24}^2 & 1 \\ a_{31}^2 & a_{32}^2 & 0 & a_{34}^2 & 1 \\ a_{41}^2 & a_{42}^2 & a_{43}^2 & 0 & 1 \end{vmatrix}$$

The vanishing of this determinant gives the relation between the six mutual distances of four coplanar points.

If we put

$$2s = a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24},$$

the sum of the products of pairs of opposite edges, we obtain

$$R^2 = \frac{s(s - a_{12}a_{34})(s - a_{23}a_{14})(s - a_{31}a_{24})}{36V^2}.$$

If the four coplanar points lie on a circle, R must be indeterminate and therefore one of the factors of the numerator vanishes, which is in effect Ptolemy's theorem.

When one of the factors of the numerator vanishes, R is indeterminate and therefore the four coplanar points lie on a circle. This is the converse of Ptolemy's theorem.

Taking $n=4$, we obtain the relation between the ten mutual distances of five arbitrary points in three-dimensional space by equating W_4 to zero. The condition that five points of three-dimensional space should lie on a sphere is then

$$|a_{\alpha\beta}^2| = 0,$$

where the indices take the values 1, 2, 3, 4, 5. This is the form which Ptolemy's theorem takes in three-dimensional Euclidean space.

L. M. MILNE-THOMSON.

THE TEACHING OF INDICES AND LOGARITHMS.

By E. V. Smith.

The Algebra Report is silent on the teaching of indices. The writer has found that some graphical illustration on the lines (probably well known) suggested below gives the work more meaning for boys of various degrees of ability.

The first step is to plot the values of 2^x, $1{\cdot}5^x$, $1{\cdot}2^x$, 1^x, and $0{\cdot}5^x$ for the values $x=1$, 2 and 3; and, assuming that we may, to join up the sets of points by smooth curves. This, of course, is done rather tentatively, as three points do not give much definition to a curve, but the values of all except 2^x when $x=4$ can be added and make the curves more definite. To include 2^4 makes the scale of the graph undesirably small.

Consideration of how the curves might be continued to the left leads to the question, "What meaning can there be in such expressions as 2^0, $0{\cdot}5^{-1}$?" The usual discussion follows of the laws applicable to positive integral indices, of our freedom to give any meanings we like to negative and fractional indices, and of the convenience of choosing such meanings as will fit with the laws already found to hold for positive integral indices. From the law

$$a^m \times a^n = a^{m+n}$$

we find that a^0 must be taken to be 1. What, then, is the value of 2^0, of $(1{\cdot}5)^0$, of $(1{\cdot}2)^0$, 1^0, and $(0{\cdot}5)^0$? Do the answers suit the graphs drawn? Insert the value and continue the graphs.

Next consider a^{-1} and a^{-n}, and show that $a^{-n}=\dfrac{1}{a^n}$.

What are the values of 2^{-1}, $(1{\cdot}5)^{-1}$, 1^{-1}, etc.? Insert these points and continue the graphs, observing that the values fit the curves very well.

As a last stage consider $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, etc., and when it has been seen that $2^{\frac{1}{2}}=\sqrt{2}\simeq 1{\cdot}4$, and $2^{\frac{3}{2}}=\sqrt{8}\simeq 2{\cdot}8$, verify that the graph does in fact give these values very nearly.

Examples on indices should be easy, and should be given few at a time and often till they are done without difficulty.

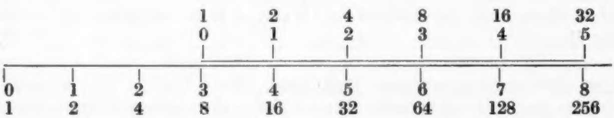
With better sets the above work may easily be done before logarithms are used at all, but less able classes may already have been taught to use the logarithms mechanically, without understanding how they work. For both kinds of class the method of obtaining logarithms by calculating and plotting fractional powers of 10 follows naturally after this work on indices, and is very useful, for the better boys as an introduction to logarithms, and for the others as an explanation of the logarithms they use, and a revision of their manipulation. The object of the work is here to show that logarithms are indices, common logarithms being powers of 10, and that the laws for manipulation depend on the laws of indices. We

are not concerned to obtain the values of the logarithms with great accuracy.

The labour involved is really very slight. The square root of 10, the square root of the result, and again a square root give the values of $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{8}}$, and products of these give $10^{\frac{3}{8}}$, $10^{\frac{5}{8}}$, $10^{\frac{3}{4}}$ and $10^{\frac{7}{8}}$. Square root tables may legitimately be used for the first three results.

The Algebra Report states that this method gives results to 2 figures (quite sufficient for the purpose), and surprisingly claims that it is more laborious than raising 1·1 to the 25th power working to 7 places, interpolating, dividing by 24·15, and finally getting results to one more figure.

The better time in teaching for the second method would seem to come when the class, understanding what logarithms are, asks how they are calculated. This almost always happens, as it should; and then it can be more appropriately pointed out that the essential fact about the logarithms is that they progress in equal steps, while the corresponding numbers progress in equal ratios, and the argument of the report developed. It will be easier for the class to



understand why the indices of 1·1, the reference numbers, are divided uniformly by 24·15; and to see that as 10 has been shown to be $(1{\cdot}1)^{24{\cdot}15}$ and 8, say, to be $(1{\cdot}1)^{21{\cdot}81}$, then

$$8 = [(1{\cdot}1)^{24{\cdot}15}]^{21{\cdot}81 \div 24{\cdot}15}$$
$$= 10^{21{\cdot}81 \div 24{\cdot}15},$$

and the result of the division gives the power to which 10 must be raised to give 8, that is, the number which has been called the logarithm of 8. If no explanation of this is given, the connection between the logarithms obtained and the base 10 must remain very obscure.

Boys find it interesting to make a paper slide rule for multiplication and division, and because it rubs in the principles of indices and logarithms, making one is of some value in teaching. The first step is to prepare two scales, which can be placed alongside one another, marked in equal divisions numbered consecutively from zero upwards. Two strips of lined paper torn off at right angles to the lines, or two strips of graph paper would serve the purpose well.

If one scale is now moved along the other till its zero comes opposite to the 3 of the other scale, then all divisions on the second scale are marked 3 more than their opposite numbers, so that the scales can be used to add numbers. The scales are then renumbered, each of the original numbers being replaced by 2 to that power,

and it is seen that the scales now multiply. The old zero of the first scale is now marked 1, and if it is moved opposite to the new 8 of the second scale then all numbers of the second scale are 8 times their opposite numbers.

Next it is explained that powers of any number (instead of 2) would work equally well, and that if we use powers of 10, then the first set of numbers marked on the scale are the common logarithms of the second set. To make the slide rule a piece of paper is folded to form an envelope open at the ends, and about an inch wide and twenty-five cm. long. Another piece slides inside it, and is seen through a long narrow window cut in the envelope. Scales identical with one another are marked on the slider and one edge of the window. It is convenient to take 10 cm. as the unit for the scales, so that the left-hand is marked 1, 10 cm. is marked 10, and 20 cm. 100. For other markings the logarithm of the number is found from tables, multiplied by 10, and the result measured from the left of the scale in cm. Thus 2 is 3·01 cm., 20 13·01 cm., 3 4·77 cm., 30 14·77 cm., and so on, from the left-hand end. As many intermediate divisions can be inserted as the boy desires.

The rule can be used for simple multiplications and divisions, and a class may be invited to discover how to get a square root from it. E. V. S.

1081. This expansion Gauss (*Rech. Arith.*, Paris, 1757, p. 431) suggests deriving by means of the exceedingly awkward and unmanageable process indicated by the formula

$$\frac{\sqrt{(1-\cos n\theta)}}{1-\cos\theta},$$

$\cos n\theta$ being previously supposed to be expanded in terms of powers of $\cos\theta$. *Quandoque bonus dormitat Homerus.*—J. J. Sylvester, *American Journal of Mathematics*, vol. 2, 1879, p. 369.

The Latin quotation is double-edged, since Gauss was not born till 1777! [Per Prof. E. T. Bell.]

1082. Throughout Japan heroic deeds are done every day in the national fight against figures. A simple addition or subtraction, performed several times over with the help of the clattering counting-rods, sends every Japanese clerk into agonies of despair.

The small shopkeeper still carries his ink and brushes in his belt and sits down ceremoniously to paint a receipt.

. . . telephone numbers containing a four are always offered at bargain prices, because " Shi ", the Japanese word for the figure four, also means " death ".—G. Stein, " Japanese Contrasts ", in the *Spectator*, October 11, 1935. [Per Mr. J. W. Stewart.]

1083. " The little squares refer to her niece, who showed her the arithmetical trick of writing figures in nine squares (I think) in such a way that on being added together in any direction they make fifteen."—S. Freud, *The Interpretation of Dreams*, p. 199. Eng. Trans. A. A. Brill. Third Eng. Edition, 1932 (Allen and Unwin). [Per Mr. H. Pfannmuller.]

THE SPHERICAL ANALOGUE OF CENTRAL FORCES.

BY J. S. TURNER.

1. If a particle is constrained to move on a sphere, the projection of its acceleration at P upon the tangent plane at P is here called its tangential acceleration. The line of action of the tangential acceleration touches a great circle; if O is a point on this great circle, the tangential acceleration at P is said to be directed to O. A tangential acceleration which is always directed to a fixed point O is called a central tangential acceleration.

A central tangential acceleration is assumed to be a continuous function of PO. The value assigned to the initial velocity must not be infinite. Infinitesimals are denoted by $\lambda(\Delta t)^n$, where λ must not be infinite.

2. *Theorem* 1. If a particle possessing a central tangential acceleration directed to O describes a spherical curve which does not pass through O, and if v is the velocity and p the perpendicular from O upon the tangent great circle at any point P of the curve,

$$(1) \qquad v \sin p = h,$$

where h is a constant.

Let P_0 be the position of the particle at time $t=0$, P its position at time t, P_0P_1, P_1P_2, P_2P_3, ... the arcs described in successive times Δt, p_0, p_1, p_2, ... the perpendiculars from O on the chords P_0P_1, P_1P_2, P_2P_3, ..., v_0, v_1, v_2, ... the velocities at P_0, P_1, P_2, ...

If the acceleration ceases when the particle arrives at P_1, then in the next interval Δt the particle will describe P_1Q along the tangent arc at P_1. If the particle is at rest at Q, and is acted on

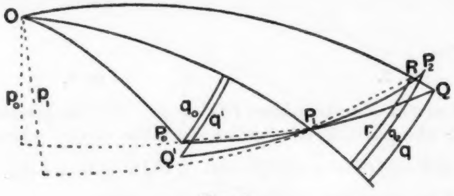


FIG. 1.

by the acceleration at Q, then in time Δt it will reach some point R on QO. Let q, r be the perpendiculars from Q, R on OP_1, then

$$\frac{\sin q - \sin r}{\sin q} = \frac{\sin OQ - \sin OR}{\sin OQ} = \frac{2\cos\frac{1}{2}(OQ+OR)\sin\frac{1}{2}QR}{\sin OQ}.$$

Now $\sin q < \sin P_1Q < v_1\Delta t$, and $QR = \frac{1}{2}f(\Delta t)^2 + \lambda(\Delta t)^3$ where f is the acceleration at Q, hence $\sin q - \sin r = \lambda(\Delta t)^3$.

In time Δt, the change in acceleration is of order Δt, both in magnitude and direction. Hence the particle, proceeding from P_1 with velocity v_1, is deflected from Q to some point P_2 on the curve such

that $QP_2 = QR + \lambda(\Delta t)^3$ and $\angle P_2QR = \lambda\,\Delta t$. Therefore $RP_2 = \lambda(\Delta t)^3$, and, q_2 being the perpendicular from P_2 on OP_1, $\sin q - \sin q_2 = \lambda(\Delta t)^3$.

Similarly, if Q' is taken on P_1Q so that $Q'P_1 = P_1Q$, and perpendiculars q_0, q' are drawn from P_0, Q' to OP_1, $\sin q' - \sin q_0 = \lambda(\Delta t)^3$. But $q' = q$, hence $\sin q_2 - \sin q_0 = \lambda(\Delta t)^3$. Therefore, from the triangles OP_0P_1, OP_1P_2,

$$\begin{aligned}\sin p_0 \sin P_0P_1 - \sin p_1 \sin P_1P_2 &= \sin q_0 \sin OP_1 - \sin q_2 \sin OP_1 \\ &= \lambda(\Delta t)^3.\end{aligned}$$

A similar equation is obtained from each successive pair of arcs. The sum of n equations, where $t = n\,\Delta t$, gives

$$\text{(2)} \qquad \sin p_0 \sin P_0P_1 - \sin p_n \sin P_nP_{n+1} = \lambda(\Delta t)^2.$$

To obtain (1), divide (2) by Δt and let $\Delta t \to 0$.

3. *Theorem* 2. A particle describes a circle of spherical radius a with uniform speed v. To find the central tangential acceleration f.

Since the acceleration of the particle is directed to the centre of the circle, the tangential acceleration is directed to the pole. Let P be the position of the particle at any instant. If the tangential acceleration does not act, then in time Δt the particle moves along

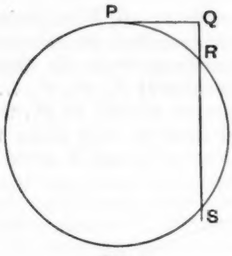


FIG. 2.

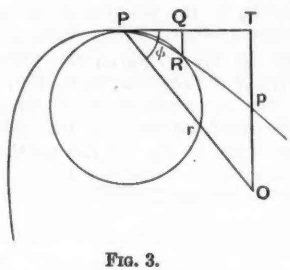


FIG. 3.

the tangent arc at P to Q, where $PQ = v\,\Delta t$. The tangential acceleration deflects the particle to a point R on the circle, where

$$QR = \tfrac{1}{2}f(\Delta t)^2 + \lambda(\Delta t)^3 \text{ and } \angle PQR = 90^\circ + \lambda\,\Delta t.$$

Produce QR to meet the circle again in S, then

$$\tan \tfrac{1}{2}QR \tan \tfrac{1}{2}QS = \tan^2 \tfrac{1}{2}PQ.$$

Hence $\tfrac{1}{4}f(\Delta t)^2 \tan \tfrac{1}{2}QS - \tfrac{1}{4}(v\,\Delta t)^2 = \lambda(\Delta t)^3$. Divide by $(\Delta t)^2$ and let $\Delta t \to 0$, then

$$\text{(3)} \qquad f = \frac{v^2}{\tan a}.$$

4. *Theorem* 3. To find an expression for the central tangential acceleration when the centre and the curve are given.

Let v be the velocity and f the central tangential acceleration at any point P of the curve, p the perpendicular from the centre O on

the tangent arc PT, r the arc OP, ϕ the angle OPT. Then the normal component of $f = f\sin\phi = f\dfrac{\sin p}{\sin r}$.

If this normal component does not act, then in time Δt the particle moves along PT to Q, where $PQ = v\,\Delta t + \frac{1}{2}f\cos\phi\,(\Delta t)^2 + \lambda(\Delta t)^3$. The normal component deflects the particle to R, where

$$QR = \tfrac{1}{2}f\sin\phi\,(\Delta t)^2 + \lambda(\Delta t)^3 \text{ and } \angle PQR = 90^\circ + \lambda\,\Delta t.$$

Describe the circle which touches PQ at P and passes through R, and proceed as in Theorem 2. Then $f\sin\phi = \dfrac{v^2}{\tan\rho}$, where ρ is the spherical radius of curvature at P.

Equate the values of $f\sin\phi$, substitute $v = \dfrac{h}{\sin p}$, and solve for f, then

$$f = \frac{h^2\sin r}{\sin^3 p\tan\rho}. \tag{4}$$

5. *Theorem* 4. To find the central tangential acceleration when a particle describes a spherical conic, the acceleration being directed to a focus.

Let rectangular axes whose origin is at the centre of the sphere meet the sphere in ξ, η, ζ. Let the centre of the conic be at ζ, its semi-major axis a along $\zeta\xi$, its semi-minor axis b along $\zeta\eta$. Let the coordinates of any point P on the sphere be $\xi = \cos P\xi$, $\eta = \cos P\eta$, $\zeta = \cos P\zeta$. Then the equation of the conic is

$$\frac{\xi^2}{\tan^2 a} + \frac{\eta^2}{\tan^2 b} = \zeta^2.$$

Let a particle describe the conic under an acceleration directed to

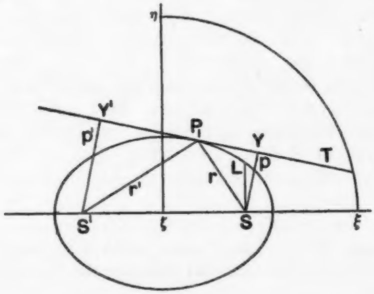


FIG. 4.

the focus $S = (\sin d, 0, \cos d)$, where $\cos d = \dfrac{\cos a}{\cos b}$. Let $P_1 = (\xi_1, \eta_1, \zeta_1)$ be any position of the particle, p the perpendicular from S on the

Y

tangent at P_1, r the focal distance SP_1, ρ the spherical radius of curvature at P_1. Then

$$\sin p = \frac{1}{k}\left(-\xi_1 \frac{\sin d}{\tan^2 a} + \zeta_1 \cos d\right) \text{ where } k^2 = \frac{\xi_1^2}{\tan^4 a} + \frac{\eta_1^2}{\tan^4 b} + \zeta_1^2,$$

$$\sin r = -\xi_1 \sin d \cot a + \zeta_1 \cos d \tan a = k \tan a \sin p,$$

$$\tan \rho = k^3 \tan^2 a \tan^2 b;$$

hence, by (4),

$$f = \frac{h^2 \tan a}{\sin^2 r \tan^2 b} = \frac{\mu}{\sin^2 SP},$$

where $\mu = \dfrac{h^2}{\tan SL} = \dfrac{h^2 \tan a}{\tan^2 b}$, SL being the semi-latus rectum.

Cor. 1. If v is the velocity at P_1,

$$v^2 = 2\mu\left(\frac{1}{\tan r} - \frac{1}{\tan 2a}\right). \tag{5}$$

For $\dfrac{\sin S'Y'}{\sin SY} = \dfrac{\sin r'}{\sin r}$, $\sin SY \sin S'Y' = \cos^2 a \tan^2 b$, hence

$$v^2 = \frac{h^2}{\sin^2 p} = \frac{h^2}{\cos^2 a \tan^2 b} \frac{\sin r'}{\sin r} = \frac{2h^2}{\sin 2a \tan SL} \frac{\sin (2a - r)}{\sin r}$$

$$= \frac{2h^2}{\tan SL}\left(\frac{1}{\tan r} - \frac{1}{\tan 2a}\right).$$

Cor. 2. To find the locus of a particle P which moves on a sphere under a central acceleration $\dfrac{\mu}{\sin^2 SP}$ directed to S.

Let v be the velocity at P_1 along TP_1, draw SY perpendicular to TP_1 and produce it to Z, making $YZ = SY$, and produce ZP_1 to S' so that $S'Z = 2a$, where $2a$ is given by (5). The spherical conic having S, S' as foci and major axis equal to $2a$ is uniquely determined, moreover it touches TP_1 at P_1. By Theorem 4 this conic is part of the required locus. If the locus contains a point Q not on the conic, then the particle, proceeding from P_1 to Q, must leave the conic at some point R; then at R the velocity must have at least three values, viz. two corresponding to each direction through R in which the conic can be described, and one corresponding to the path RQ. Now the velocity at R is given by (5) where $r = SR$, hence it can assume only two values. Therefore this conic is the entire locus of P.

J. S. T.

1084. TORRICELLI (Evangelista) OPERE, edite in Occasione del III Centenario della Nascita, col Concorso del Comune di Faenza da Gino Loria e Giuseppe Vassura.

The work was privately printed at the expense of the municipality of Faenza, one of the many instances foreign countries show in honouring their illustrious sons, while no British authority, high or low, has been found to take an interest in producing the works of the greatest scientist of all—Sir Isaac Newton.—Sotheran's *Price Current*, No. 843, p. 19.

A METHOD OF LONG DIVISION FOR SMALL DIVISORS.

BY A. A. FLETCHER-JONES.

THE method is best introduced by a simple example.

To work out $\frac{1}{19}$ as a recurring decimal.

Instead of long division by 19, proceed as follows:

$$\begin{array}{l}
2\,)\,\underline{1{\cdot}0} \\
\begin{array}{cccccccccccccccccc}
 & & ^1 & & & ^1 & ^1 & ^1 & ^1 & & ^1 & & ^1 & ^1 & & & & \\
0{\cdot}0 & 5 & 2 & 6 & 3 & 1 & 5 & 7 & 8 & 9 & 4 & 7 & 3 & 6 & 8 & 4 & 2 & 1, \text{ etc., etc.}
\end{array}
\end{array}$$

[*Explanation.* Divide 2 into 1·0 and write the quotient 5 one decimal place further on than usual.

Now divide 2 into the resulting quotient 5, and obtain the next decimal figure 2, with remainder 1. This remainder 1 is written above the quotient 2, and is combined with the 2 to form the next dividend, 12.

Now divide 2 into 12 : quotient 6, remainder 0.

,,	,,	6,	,,	3,	,,	0.
,,	,,	3,	,,	1,	,,	1.
,,	,,	11,	,,	5,	,,	1.
,,	,,	15,	,,	7,	,,	1, etc., etc.

If this process is continued the first 18 digits recur. With practice there is no need to write in the remainders. I have written them above only to illustrate the method. This explanation seems cumbersome, but the process itself is very simple.]

The process is justified by the Binomial Expansion of $(1-\frac{1}{20})^{-1}$.

If this process of continued division by 2 be called "rividing" by 2, the following general result holds for any value of x.

To work out $\dfrac{1}{10x+9}$ *as a decimal, rivide by* $(x+1)$.

Long division into any number may be carried out in a similar way. The only new feature is that the figure above each successive quotient is added to that quotient to form (combined with the last remainder) the next dividend.

Example. To divide 12345 by 29.

"Rivide" by 3.

$$\begin{array}{l}
3\,)\,1\ 2\ 3\ 4\ 5 \\
\qquad\ \ \underline{\overset{7}{4}\ \overset{16}{2}\ \overset{20}{5}\cdot\overset{2}{6}\ \overset{2}{8}\ \overset{1}{9}}, \text{ etc., etc.}
\end{array}$$

[*Explanation.* Divide 3 into 12; quotient 4, remainder 0. Add the quotient 4 and the figure 3 (above it) to give next dividend, 7.

Divide 3 into 7; quotient 2, remainder 1.

Add quotient 2 to 4 (above it), combine with the last remainder 1 to obtain the next dividend 16, etc., etc.]

For divisors whose unit's digit is not 9, the process is more complicated, but in many instances still much quicker than long division.

Again an example, $\frac{1}{78}$.

$$\begin{array}{l} 8\ \underline{)\ 1{\cdot}0} \quad {\scriptstyle 22\ 64\ 16\ 4\ 40\ 10\ 22\ 64} \\ \quad\ 0{\cdot}0\ 1\ 2\ 8\ 2\ 0\ 5\ 1\ 2, \text{ etc. (the figures recurring).} \end{array}$$

[*Explanation.* The successive dividends are written above each digit. Start as before "rividing" by 8. The first quotient is 1, remainder 2. The next dividend, 22, is obtained by combining the remainder 2 with *twice the quotient*, 1.

Divide 22 by 8; quotient 2, remainder 6.

The next dividend, 64, is now obtained by combining the remainder 6 with *twice* the quotient 2, etc., etc.]

In fact the process is a successive division by 8 and alternate multiplication of quotients by 2. If this be called "rividing" by 8 and "multividing" by 2, the following general result holds:

To divide by $(10x+y)$, *rivide by* $(x+1)$ *and multivide by* $(10-y)$.

Long division into any number can be performed by a similar method to that employed when dividing by $(10x+9)$ above.

There is an unfortunate complication to the method, which again will be illustrated with an example, $\frac{1}{28}$. Rivide by 3 and multivide by 2.

$$\begin{array}{l} 3\ \underline{)\ 1{\cdot}0} \quad {\scriptstyle 16\ 20\ 32\ 40\ 36\ 24\ 16\ 20} \\ \quad\ 0{\cdot}0\ 3\ 5\ 6\ 0\ 3\ 2\ 8\ 5, \text{ etc.} \\ \qquad\qquad\ \ \underline{1\ 1\ 1\qquad\quad} \\ \quad\ 0{\cdot}0\ 3\ 5\ 7\ 1\ 4\ 2\ 8\ 5, \text{ etc.} \end{array}$$

[*Explanation.* All goes well until the digit 6 is reached. The remainder 2 when combined with twice the quotient 6 gives 32 as the next dividend. So the next quotient is 10, and the 1 has to be carried back to add to the preceding digit 6.

The next dividend, 40, is obtained by combining the remainder 2 with twice the quotient 10, etc., etc.]

This complication, whenever it occurs, makes the process cumbersome, and for divisors with a small unit's digit, the method is of doubtful practical value.

I have seen no other work on this subject, but perhaps some reader may know of some previous discovery of this process.

A. A. F.-J.

1085. "What's the average tip they give you fellows?"

"'Bout a dollar, Sah."

"There you are then, but it seems rather a lot."

"Dat's jest grand, Sah. Yo' de very first man to come up to de average." —*Punch*, June 5, 1935, p. 679.

1086. Already her broad white wake has finished with the great circle track, and she is drawing a line straight for the Ambrose light vessel.—*Observer*, May 31, 1936. [Per Prof. E. H. Neville.]

MATHEMATICAL NOTES.

1213. *American tournaments.*

$2n$ teams are to play an American tournament, n matches taking place simultaneously. It is required to find a simple method of arranging the matches.

Represent the teams by letters and arrange $2n-1$ of them in the form of a regular polygon, with the remaining one in the centre. Starting with any side of the polygon, draw a series of parallels connecting the vertices in pairs. One vertex is left over and this is joined to the centre. There are now n lines indicating one set of n matches. The other sets are found similarly, using the different sides of the polygon in turn.

For $2n-1$ teams, omit the letter in the centre.

E. H. LOCKWOOD.

1214. *An elementary note in Trigonometry.*

Many a beginner must have wondered whether there are any angles besides the familiar ones which are expressible simply in degrees and have simple trigonometrical functions. It is known that there are none; but it is useless to refer a beginner to theoretical discussions of the binomial equation $x^n=1$. It may therefore be worth while to point out that the formula for $\cos 2A$ to some extent answers the question.

If the ratio of an angle $A°$ to four right angles is a rational number, that is to say, if $A/360=m/n$ where m and n are integers having no common factor, the points on the unit circle having vectorial angles

$$A, 2A, 3A, 4A, \ldots \textit{ ad inf.}$$

lie at the angular points of a regular polygon of n sides. The cosines of these angles, therefore, have only a finite number of values, and in any infinite sequence of these angles the values of the cosines must after a time be repeated.

On the other hand, if B is an angle whose cosine is a rational number, that is to say, if $\cos B=p/q$ where p and q are integers having no common factor, then

$$\cos 2B=(2p^2-q^2)/q^2.$$

No cancelling is possible if q is odd, but if q is even and p therefore odd a factor 2 may be thrown away; thus $\cos 2B$ is a rational number with a denominator either q^2 or $\frac{1}{2}q^2$, in either case greater than that of $\cos B$ (the familiar cases of $q=1$ or 2 and $p=\pm 1$ being excluded). So again, $\cos 4B$ is a rational number with a still larger denominator. In fact, the values of

$$\cos B, \cos 2B, \cos 4B, \cos 8B, \cos 16B, \ldots \textit{ ad inf.}$$

are a sequence of rational fractions with constantly increasing denominators; no value can be repeated; therefore the angle B cannot be one of the angles A.

Further, if the sine, or the cosine, or the tangent of an angle C

is either a rational number or the square root of such a number, $\cos 2C$ is rational: $2C$ and therefore C cannot be included among the angles A. The only exceptions are multiples of 30° or 45°.

I am not so rash as to assert that this argument is new; it is improbable that it has not been noticed. HERBERT W. RICHMOND.

1215. *The Pythagorean triangle and its analogues.*

For convenience of reference in what follows, the three cases in which an angle commensurable with 180° is involved in a triangle with commensurable sides are given below (m and n integers, $m>n$).

Angle C	c^2	a	b	c
(i) 60° - -	a^2-ab+b^2	$2mn-n^2$	m^2-n^2	m^2-mn+n^2
(ii) 90° - -	a^2+b^2	$2mn$	m^2-n^2	m^2+n^2
(iii) 120° - -	a^2+ab+b^2	$2mn+n^2$	m^2-n^2	m^2+mn+n^2

Within the limits 0° and 180° the only angles commensurable with 180° having rational cosines are known to be 60°, 90°, and 120°; see the note by Dr. H. W. Richmond above. It follows that no other cases than the three in the above table are possible.

In Dickson's *History of the Theory of Numbers* there is reference to a note on (i) by J. Neuberg (1874) and to a proof of (iii) by Neuberg and G. B. Mathews (1887).

An interesting connection between (iii) and (ii) was revealed by Mr. J. P. McCarthy in his solution of the problem of finding three Pythagorean triangles equal in area; see Note 1196 (May, 1936). A similar relation might be expected to hold between (i) and (ii), and this is the case, provided that the right-angled triangles are formed from (c, a), (c, b) and $(c, |a-b|)$. The area is then $abc\,|a-b|$. It will be found that for a given value of n, to obtain the same set as that given by (iii),

$$m_{\text{I}} = m_{\text{III}} + n.$$

This relation suggests transforming the functions of m and n in (i), (ii) and (iii) by the substitution of $m \pm n$ for m. If we take $m+n$, the new forms contain only positive terms, thus removing the restriction $m > n$:

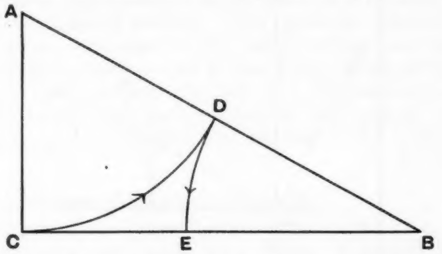
	a	b	c	$a+b-c$
(i) -	$2mn+n^2$	m^2+2mn	$m^2+n(m+n)$	$3mn$
(ii) -	$2mn+2n^2$	m^2+2mn	$m^2+2n(m+n)$	$2mn$
(iii) -	$2mn+3n^2$	m^2+2mn	$m^2+3n(m+n)$	mn

To obtain triads whose members are prime to each other

in (i) $|m-n|$ must not be a multiple of 3 ;
in (ii) m must not be a multiple of 2 ;
in (iii) m must not be a multiple of 3.

m and n may both be odd and of course must be prime to each other. Thus the first five prime triads of (ii) are given by (1, 1), (1, 2), (3, 1), (1, 3), (3, 2).

The ratio m/n involved in a given triad may be shown in the case of (ii) by the construction below : $AD = AC$, $BE = ED$.



In this form, $m/n = CE/EB$; in the classical form, $m/n = CB/EB$.

It will be seen that if, using either form, the series (i), (ii), (iii) be extended one step backward, the analogous conditions for $C = 0°$ are given ; if one step forward, for $C = 180°$.

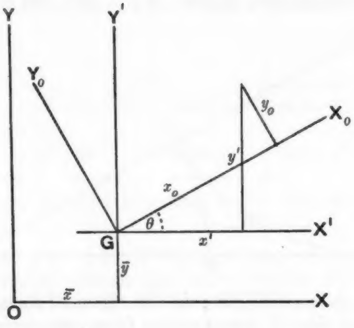
A. N. FITZGERALD.

1216. *On rolling and the instantaneous centre of rotation.*

What is the definition of rolling ? The Third Form arithmetician is easily convinced that the number of revolutions made by a bicycle wheel 28 inches in diameter as the cyclist travels one mile is the ratio of that distance to the circumference of the wheel. He in fact accepts equality of arc on the moving and stationary curves as the criterion of rolling. Later, friction is recognized as a passive reaction tending to prevent relative motion of the particles in contact and its success distinguishes rolling from sliding. It now appears that the statement " A rolls on B " is tantamount to the statement " the particle of A which is in contact with B is, at the instant, at rest ". If we accepted this as the definition of rolling it would be unnecessary to prove the theorem that the plane motion of a lamina consists of the rolling of the body-locus of the instantaneous centre on the space-locus of that point. For by the definition of these loci, from instant to instant corresponding points of the curves coincide and the point of coincidence is the point of the lamina which is at rest at the instant. What the theorem proves is that corresponding arcs of the two loci are of equal length and the proof usually makes a strong appeal to intuition, treating the curves as the limits of polygons, of which the moving one undergoes a

sequence of finite rotations (*see* Ramsey's *Dynamics*, p. 60). Adopting equality of arc as the definition of rolling, Ramsey proves the fact of rolling. In their *Mechanics* Palmer and Snell give an analytical proof of the converse, namely, that given the fact of rolling (defined by equality of arc) it follows that the point of contact is at rest.

The following is an analytical proof of Ramsey's proposition, which may be put in the form : if s_0 and s denote corresponding arcs of the body-locus and space-locus of the instantaneous centre, to prove that $ds_0 = ds$.



OX, OY are fixed axes.

GX', GY' are parallel axes through a point G which is fixed in the body.

GX_0, GY_0 are axes fixed in the body.

(x, y), (x', y') and (x_0, y_0) are the coordinates of I referred to these three sets of axes, $(\bar{x}, \bar{y})$ are the coordinates of G referred to OX, OY.

Let (u, v) be the component velocities of G in the directions of OX, OY and ω the angular velocity of the lamina. Then, since the velocity of I is zero

$$u - \omega y' = 0, \quad \text{and} \quad v + \omega x' = 0.$$

For the space-locus of I,

$$x = \bar{x} + x' \quad \text{and} \quad y = \bar{y} + y'.$$

Thus
$$\begin{aligned} dx &= d\bar{x} + dx' \\ &= u\,dt + dx' \\ &= y'\omega\,dt + dx' \\ &= y'\,d\theta + dx', \end{aligned}$$

and similarly
$$dy = -x'\,d\theta + dy'.$$

For the body-locus,

$$x_0 = x' \cos\theta + y' \sin\theta,$$

and
$$y_0 = -x' \sin\theta + y' \cos\theta,$$

and $$dx_0 = dx' \cos\theta - x' \sin\theta d\theta + dy' \sin\theta + y' \cos\theta d\theta$$
$$= \cos\theta(dx' + y'd\theta) + \sin\theta(-x'd\theta + dy')$$
$$= \cos\theta dx + \sin\theta dy.$$

Similarly, $$dy_0 = -\sin\theta dx + \cos\theta dy.$$

Squaring and adding,

$$dx_0^2 + dy_0^2 = dx^2 + dy^2,$$

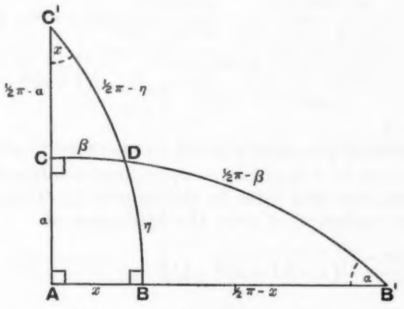
or $$ds_0^2 = ds^2.$$ Q.E.D.

Either this or the converse proposition may be used to reconcile the two definitions of rolling. The question remains: when in dynamics we are concerned with a wheel rolling on a straight road, should we deduce the relation $v = a\omega$ from the equality of arc and distance travelled, or from the fact that the point of contact is at rest? The latter alternative (that is, the second definition of rolling) seems more consistent with the rest of our work on friction.

C. G. PARADINE.

1217. *The right-angled spherical triangle.*

The relation given below between the parts of a right-angled spherical triangle can easily be verified. I should be interested to know if it is a simple deduction from some general geometry on the sphere.



Let $ABDC$ be a spherical quadrilateral, right-angled at A, B, C; let $x = AB$, $\eta = BD$, $a = AC$, $\beta = CD$; also let $\tanh \frac{1}{2}y = \tan \frac{1}{2}\eta$, $\tanh \frac{1}{2}b = \tan \frac{1}{2}\beta$; then

$$\cos(x + iy)\cos(a - ib) = 1.$$

This can be read as a relation between the sides of the quadrilateral, or as a relation between the parts of either of the right-angled triangles $BB'D$, $CC'D$, and hence of any right-angled triangle.

F. BOWMAN.

1218. *A probability problem.*

From a pack of cards containing m suits of n cards each, numbered 1, 2, ... n, s cards are taken at random. These cards are found to contain p different numbers. To find the average value of p.

The number of different ways of selecting s cards from the pack of mn is

$$ {}_{mn}C_s. \qquad \text{......(i)}$$

We begin by finding the number of ways of choosing s cards from the pack to contain a given number p_0, of different numbers ($p_0 \leqslant n, s$). There are m cards of each particular number. Hence the number of ways of selecting q cards each numbered 1 is ${}_mc_q$. It follows that the required number of ways of choosing s cards to contain p_0 different numbers is equal to the coefficient of x^s in the expansion of

$$({}_mC_1x + {}_mC_2x^2 + \ldots + x^m)^{p_0} = \{(1+x)^m - 1\}^{p_0}. \qquad \text{......(ii)}$$

Now each of the ways of selecting p_0 different numbers from n gives rise to a new set of selections given by (ii).

Hence the required average value, remembering (i), is the coefficient of x^s in the expansion of

$$\sum_{p=1}^{\min(n,s)} \frac{p \cdot {}_nC_p}{{}_{mn}C_s}\{(1+x)^m - 1\}^p$$

$$= \frac{1}{{}_{mn}C_s} \sum_{p=1}^{\min(n,s)} \frac{p \, . \, n!}{p!\,(n-p)!}\{(1+x)^m - 1\}^p$$

$$= \frac{n}{{}_{mn}C_s} \sum_{p=1}^{\min(n,s)} \frac{(n-1)!}{(p-1)!\,(n-p)!}\{(1+x)^m - 1\}^p,$$

$$= \frac{n\{(1+x)^m - 1\}}{{}_{mn}C_s} \sum_{p=1}^{\min(n,s)} {}_{n-1}C_{p-1}\{(1+x)^m - 1\}^p,$$

where ${}_{n-1}C_0 = 1$.

If $n > s$, the least power of x in $\{(1+x)^m - 1\}^{s+1}$ is x^{s+1}. Hence we may take the sum to n in either case, without altering the coefficient of x^s. Further, the first term in the sum is 1. Thus the required average is the coefficient of x^s in the expansion of

$$\frac{n\{(1+x)^m - 1\}}{{}_{mn}C_s}[1 + \{(1+x)^m - 1\}]^{n-1}$$

$$= n(1+x)^{mn-m}\{(1+x)^m - 1\}/{}_{mn}C_s$$

$$= n\{(1+x)^{mn} - (1+x)^{mn-m}\}/{}_{mn}C_s.$$

Thus the average value of p is

$$n({}_{mn}C_s - {}_{mn-m}C_s)/{}_{mn}C_s$$

$$= n - n\frac{{}_{mn-m}C_s}{{}_{mn}C_s}.$$

For a selection of 13 cards from an ordinary pack of 52, we get $s = 13$, $n = 13$, $m = 4$.

Thus
$$\bar{p} = 13 - 13 \frac{{}_{48}C_{13}}{{}_{52}C_{13}}$$
$$= 13 - 13 \frac{48!}{13!\,35!} \cdot \frac{13!\,39!}{52!}$$
$$= 13 - 13 \frac{39 \cdot 38 \cdot 37 \cdot 36}{52 \cdot 51 \cdot 50 \cdot 49}$$
$$= 9{\cdot}05\ldots .$$

G. A. GARREAU.

1219. *A short solution of the plane wave equation* $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$.

The formal solution of the plane wave equation $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$ is usually a rather protracted affair, so the following may be of interest:

If $\partial/\partial t = p$ (Heaviside's operator), then $\dfrac{d^2\phi}{dx^2} - \dfrac{p^2}{c^2}\phi = 0$, of which the operational solution is

$$\phi = A_1 e^{-px/c} + A_2 e^{px/c} = \phi_1 + \phi_2. \qquad \ldots\ldots(1)$$

Assume $\phi_1 = f_1(t)$ when $x = 0$, then $A_1 = f_1(t)$, and according to operational rules

$$\phi_1 = e^{-px/c} f_1(t) = f_1(t - x/c). \qquad \ldots\ldots(2)$$

In like manner we get $\phi_2 = f_2(t + x/c)$, so

$$\phi = f_1(t - x/c) + f_2(t + x/c). \qquad \ldots\ldots(3)$$

Alternatively $\phi_1 = e^{-px/c} f_1(t)$, and by the Bromwich rule

$$\phi_1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z(t-x/c)}\,\psi(z)\,\frac{dz}{z}, \qquad \ldots\ldots(4)$$

where $\psi(p)$ is the operational form of $f_1(t)$, which means that

$$f_1(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\,\psi(z)\,\frac{dz}{z}, \qquad \ldots\ldots(5)$$

Hence
$$f_1(t - x/c) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z(t-x/c)}\,\psi(z)\,\frac{dz}{z}, \qquad \ldots\ldots(6)$$
$$= \phi_1. \qquad \ldots\ldots(7)$$

The second solution can be found in like manner.

N. W. MCLACHLAN.

1087. "I measured you. It's like astronomy. An astronomer wants to get the orbit of a star. He takes its position twice—and from the two observations he can calculate the orbit to the inch. I've got three observations of your orbit. Enough—and to spare."

"I shan't misunderstand again," said Susan.—David Graham Phillips, *Susan Lenox, her fall and rise*, Vol. II, p. 240 (Appleton, New York, 1931). [Per Mr. Frank Robbins.]

REVIEWS.

The Nature of Physical Theory. By P. W. Bridgman, Hollis Professor of Mathematics and Natural Philosophy, Harvard University. Pp. 138. 9s. 1936. (Princeton University Press; Humphrey Milford, Oxford University Press)

In this fascinating, stimulating and exasperating volume Professor Bridgman gives in expanded form the substance of three lectures given at Harvard in December, 1935. Professor Bridgman writes explicitly as an experimental physicist, and disclaims any intention of having anything to say to the technical logician. Whether he is of the opinion that the technical logician (with whom I am anxious to class the mathematician, though this would horrify Professor Bridgman) is already in a fair way to master the author's difficulties is not made clear, but Professor Bridgman is desirous of enquiring how far the phenomena of physics may be "understood", in the awakening awareness coming to the physicist. He distinguishes between the critical and theoretical activities of the physicist. The task of the theoretical physicist is to compress all experimental knowledge into an understandable point of view, and he would regard his last and most successful theory as a structure of limited validity, always subject to the necessity for radical alteration: the task of the physicist as critic is to map out the possibilities and limitations of the human mind, a task which once learned will be no more needed. "If all the theories presented to our future physicist for observation are constructed on a basis of sound criticism, he will acquire instinctively and without conscious effort the art of properly constructing his theories." Professor Bridgman then very rightly proceeds to explain what he means by "properly constructed". He emphasizes the importance of the description of a situation in terms of what he calls "operations", or, as it is perhaps more frequently expressed, the method of defining "meaning" in terms of "observables", a method due perhaps in principle to Locke and Hume but revived as a tool of research in modern times by Einstein and Heisenberg. In brief we must say exactly not what we *think* but what we *do*, in arriving at a description of a situation. Thus "our theories reduce in the last analysis to descriptions of operations actually carried out in actual situations, and so cannot involve us in any inconsistency or contradiction. Thus is solved at one stroke the problem of constructing our fundamental physical concepts so that we shall never have to revise them in the light of later experience". It is unfortunate that in this analysis the author unconsciously passes from "theories" to "concepts". But the isolating of a concept and the construction of a theory are not the same thing. "Concepts" (as, for example, the concept of a "particle", or the concept of "energy" in some presentations of thermodynamics) are things difficult to define with precision but tacitly accepted as a basis of understanding between writer and reader, whilst "theories" are roughly the relations between the concepts, and the whole point of mathematical physics is the problem of ascertaining to what extent we can identify in the outside world entities *realizing* or approximately corresponding to the concepts as judged by their possession of relationships similar to those yielded by the theory.

That we must say exactly what we *mean* is the corner-stone not only of modern physical theories such as relativity and quantum mechanics, but also of disciplines such as modern geometry or modern analysis. Professor Bridgman would apparently subscribe to this in his oft-repeated but insufficiently argued statement that "mathematics is an experimental science". If so, mathematics is on the same footing as any other so-called "exact" science. That certain elements derived from experience guide us in the construction of

mathematics one would be loath to deny, but this is a very different thing from saying that mathematics is an experimental science. As this misunderstanding lies at the very root of Professor Bridgman's difficulties, it is worth while to pursue it a little further. Physical science appears to contain certain " brute facts ". Let us take a mathematical brute fact—that the three perpendiculars from the vertices to the sides of a plane triangle meet in a point. The Egyptians, who had some observational knowledge of geometry, may well have been familiar with this fact, and, if so, to them it would be what they would call a " law of nature ", an inductive generalization based on empirical experience. Was it so to the Greeks ? Is it so to us ? I know of no memoir displaying the tabulated results of experiments with triangles and their perpendiculars. Why then did the Greeks believe it ? Why do readers of this *Gazette* believe it ? They believe it because it is a deductive inference from axiomatic definitions, not an inductive inference, of uncertain exactitude, derived by some application of the Principle of Induction. I would even deny that Euclid's " parallel postulate " is experimental ; it is merely the axiom which completes the operational definition of a Euclidean plane. The axioms of geometry do not ascribe properties to geometrical elements as " brute facts " ; they simply achieve a complete but minimal description of the subject of discourse. Thus the essence of mathematics is that it contains no brute facts, no " hypotheses " (save in genuinely experimental investigations as, for example, the Riemann hypothesis in the theory of the ζ-function), no concepts (the concept of a " point " is a most useful help to the imagination, but neither Euclid nor any other geometer explicitly appeals to this concept in deducing theorems).

We see then that Egyptian " laws of nature " become Greek " theorems ". If Professor Bridgman should argue that the concurrence of perpendiculars is not a law of nature as he understands one, but a mathematical result, then he has conceded at once that there is some essential difference between mathematics and present experimental sciences. If he wishes to maintain that mathematics is an experimental science, he must show that the security of our belief in the facts of geometry depends on careful experimentation. The actual position is that no one dreams of putting a new geometrical theorem to the test of experience—it is put to the test of man's ability to find a flaw in its " proof ". What the Greeks contributed to science was the discovery of new deductive processes, new methods of inference, new weapons of research by pure thought. The discovery of a new deductive chain of thought is just as important as, and usually more difficult than, the discovery of a new brute fact, and the deductive connections between brute facts are an essential element in science. Man as thinker is just as important as man as observer. Man observed the facts behind the first two laws of thermodynamics, but was led to the discovery of entropy by adding to these an original deductive process. Similarly, Einstein's special relativity was the consequence of a new mode of arguing. The revulsion from the Aristotelian or *a priori* method to the Baconian or inductive method has led to the under-estimation of the importance in present-day science of the discovery of new thought-processes.

Now though Professor Bridgman has much to say about " laws of nature " —whether they are covariant, and so on—he never says what a law of nature is. It is clear that laws of nature which are inductive generalizations from brute facts of observation cannot be known to be " exact " ; they cannot be known to hold good at distant places or at earlier times ; it cannot be seen why they are true. Such laws of nature are appropriate to a science in its Egyptian phase. But the whole march of scientific thought is to transform the Egyptian phase into a Greek phase, to deduce the more and more from the less and less, until peradventure we deduce the whole from the mere

axiomatic definitions which are the minimal descriptions by which we recognize what we are talking about. Geometry has achieved this level. Dynamics is on the way to it—witness Einstein's unification of the distinct concepts of mass and energy. The necessity, the urge to gain an understanding of a law of nature, is relevant only to a science at the Egyptian level. We no longer ask *why* the three perpendiculars in a triangle meet in a point; we do not seek to *understand* this brute fact; we are content when we have deduced it from more intuitively "obvious" facts in the first instance, ultimately as a logical consequence of the axioms specifying the subject of investigation.

This means that there are ultimately no brute facts—only the possibilities and limitations of the human mind as an instrument of inference, as Professor Bridgman says at the outset. We cannot establish this; we can only hold it by an act of faith. The alternative is that laws of nature are in some way the operations of magic, to believe in which is superstition. The circumstance that we have indefinitely far to travel before we reduce the chaos of facts to a deductively satisfactory order, but that we have started on the road, is to the mathematical physicist one of the charms of the journey. But he must be bold enough at the outset, whilst admitting that our *knowledge* of the external world is derived from observation, to deny that any *understanding* of the external world can be acquired in the same manner. To argue that mathematics is an experimental science is to be involved in contradictions at the beginning.

Professor Bridgman has many illuminating things to say about probability, wave mechanics and the theory of relativity. His scepticism about the validity of the bases of "general" relativity and his distinction between that and "special" relativity appeal strongly to the reviewer. But the author is less profound than usual when he scoffs at attempts to describe the whole universe mathematically. This lack of confidence is a natural result of his failure to recognize the difference between an empirical science and a mathematical science. Once it is realized that the aim of mathematical physics is to begin with operational definitions, then later to identify the relations theoretically reached with relations observed in the accessible part of the universe, there is nothing more ambitious in attempting to construct a dynamics of the entire universe than in constructing the geometry of the entire Euclidean plane. Geometry has no principles, only axioms and theorems. May we not say also that Nature has no principles—is in fact entirely unprincipled? E. A. M.

Theoretical Astrophysics. Atomic theory and the analysis of stellar atmospheres and envelopes. By S. Rosseland. Pp. xix, 355. 25s. 1936. International series of monographs on physics. (Oxford)

The latest addition to this series of monographs well maintains its international character. The author is Professor of Astronomy in the University of Oslo. Also he is founder and Director of the Institute of Theoretical Astrophysics in that university. The same broad view of the future of astrophysical science which led to the inception of this institute has led also to the planning of the present book.

The first few chapters give a short but comprehensive and logically argued account of atomic theory, which forms the foundation of all that follows. It is treated from the standpoint of wave mechanics, approached from classical analytical dynamics by way of statistical mechanics, and is developed as far as the theory of multiplets in atomic spectra. Then follows a short account of the quantum theory of radiation, with special reference to the theory of the widths of spectral lines. "Forbidden" transitions are also treated later

in the book, as far as their theory is required for astrophysical applications; and a summary of the quantum theory of diatomic molecules is given in the same way.

The central part of the book, some 150 pages, is naturally devoted to the theory of stellar atmospheres. It starts with the theory of the propagation of radiation in an atmosphere and of the formation of the continuous spectrum. The treatment of the profiles of absorption lines which follows makes a special feature of allowing from the outset for cyclical transitions between more than two atomic states. Recent work has shown the importance of doing this, especially in regard to the troublesome question of central intensities of absorption lines. The theory of thermal excitation and ionization in stellar atmospheres, and their bearing on the variation of line intensities from star to star, are there considered. It is this theory which gives a rational interpretation of the spectral classification of stars. Another difficult problem which interacts with it is that of the opacity of the atmospheres, which is next discussed. Then the influence of stellar rotation, of magnetic fields, and of inter-atomic electric fields, on the spectra are dealt with. Finally, the occurrence of molecular bands in stellar spectra, and the information they yield about the temperature and pressure in the atmosphere of stars (and incidentally of planets) is considered.

The remaining five chapters deal with stellar envelopes, by which is meant all stellar material outside the normal atmospheres which are responsible for the main features of the spectrum. The various existing theories of the solar chromosphere and corona are sketched. Then the envelopes of the stars, particularly giant stars, are considered with reference to the peculiarities they impress upon the spectra. Most notable of these are the bright lines, of which in fact Professor Rosseland himself first gave an adequate explanation in terms of cyclical transitions under the conditions sufficiently far removed from thermodynamic equilibrium which occur in such envelopes. In some cases these envelopes appear to be streaming away from the parent star, and probably in this way give rise to the gaseous nebulae, which next claim attention on account of fresh problems of ionization and excitation associated with them. Amongst these is the novel feature of " forbidden " lines. The nebulae themselves merge into the general cosmic cloud about which it is shown to be now possible to know a good deal.

Theoretical astrophysics is a difficult subject in which to begin research. To start with, the worker must be equipped with knowledge of a wider field of mathematical physics than the average researcher in mathematical physics itself, who usually is permitted to confine his attention to a single branch of the subject. Professor Rosseland provides admirably for this in his early chapters, which not only themselves contain a great many results needed in astrophysics, but give also sufficiently general mathematical methods to put the reader in a position to profit by standard works of reference on the subject.

This is followed up in the astrophysical chapters by again sketching very general mathematical methods, which should enable the student to pass on to the most recent original papers with every chance of understanding their technique. With evidently the same end in view, on the physical side the author pays more attention to laying bare the essential principles and the implicit assumptions or restrictions than to a detailed consideration of results. So much is this the case that some sections appear seriously to lack clinching conclusions. But Professor Rosseland calls the book a " programme of theoretical astrophysics " ; the implication is that the results so far obtained are but firstfruits and that the full harvest of mature results remains to be gathered. It is an encouraging view for those who work in this field.

The book is not without copious references to the literature, and a commendable feature is the frequent care to refer to the originators of ideas employed. But perhaps even a little more could have been done in the way of systematic bibliographies. For example, though the work of Fowler and Milne is quoted in the chapter on " Thermal Excitation ", Milne's more recent extensions of the work are not mentioned there. In the chapter on " The Opacity ", a treatment by Chandrasekhar rather analogous to that given is not referred to. Again, in the work on nebular luminosity, the contributions due to Zanstra scarcely receive adequate recognition; the computations by Chandrasekhar which *are* quoted were admittedly just a quantitative working out of Zanstra's ideas. One mentions this minor criticism because in every other way the book will be so tremendously useful to those starting research in astrophysics that it seems a pity that it does not always give a little more guidance as to where to look for some of the more detailed developments of certain parts of the subject. W. H. McCrea.

Statistical Research Memoirs. Vol. I. Edited by J. Neyman and E. S. Pearson. Pp. 161. 15s. 1936. (Department of Statistics, University College, London)

Until his retirement three years ago the late Professor Karl Pearson edited, in addition to *Biometrika*, a succession of publications on a very varied range of subjects. These covered a period of thirty years, being issued at first from the Department of Applied Mathematics of University College, London, and later from the combined Biometric and Eugenic Laboratories. In the year of Karl Pearson's death, his son, Professor and Head of the Department of Statistics, has collaborated with Dr. J. Neyman, Reader in the Department, in producing a volume of research memoirs, and it is their intention that a similar volume should appear each year. This series will take the place of the earlier series and will be narrower in scope, owing to the fact that the editors are responsible for teaching and research in Statistics only; R. A. Fisher, Galton Professor, is now Head of the Department of Eugenics.

The editors feel that in spite of the existence of a large number of special problems for which perfect solutions exist, statistical theory in general in its present state is far from being completely satisfactory from the point of view of its accuracy. The establishment of a theory of statistics on a level of accuracy usual in other branches of mathematics is their aim, while the *Statistical Research Memoirs* are to be the medium of publication for papers prepared under their direction; in fact, they are to be restricted entirely to papers by workers in the department. Not everyone will agree as to the desirability of such restricted departmental journals, but it is certainly convenient to know that most of the publications of this department will be found in one place. It does mean, however, that the material published will have to be very carefully studied before its level in relation to the work of other authors can be determined. Statistical research within the department will be a " sheltered industry ", and it will not be altogether easy to judge its value by comparison with the papers that compete for publication in open journals. In other words, the *Memoirs* will have to make their own reputation.

How far this is achieved in the first volume may be judged from the contents, which deal with the theory of testing statistical hypotheses. There are two papers by Neyman and Pearson which are along familiar lines, a further paper of which Neyman is a joint-author, and four others. The main papers are somewhat didactic in tone, and there is considerable repetition; it seems, in fact, as if the attempt is being made to expound a rigid theory of testing hypotheses with the aid of the higher mathematics. This is taking the theor-

etical side of statistics further and further away from the comprehension of the practical man who has to make use of statistical formulas and tests. However, the practical application of the tests is not lost sight of, and a number of the papers are concerned with these.

The *Memoirs* are to be welcomed as an interesting addition to statistical literature; they evidently merit careful study, but it is too soon yet to appraise the quality of the work turned out. Time will no doubt tell whether the early chapters of the standard treatise on mathematical statistics which yet remains to be written are likely in their definitions and approach to the philosophy of the subject to follow the Neyman-Pearsonian lines or not.

J. W.

The Rational Quartic Curve in Space of Three and Four Dimensions. By H. G. Telling. Pp. viii, 78. 5s. 1936. Cambridge Tracts, 34. (Cambridge)

The twisted cubic is a curve whose properties are familiar to all geometers, but its four-dimensional analogue is not so well known. The first half of this monograph deals in a succinct way with the geometry of the rational quartic curve in four dimensions and of the various loci associated with it, while the second part deals with the projection of this curve into ordinary space.

The early sections deal with the more elementary properties of the normal curve C, with its parametric equations and projective generations. Then the invariant quadric and cubic primals associated with the curve are introduced. The quadric contains the tangents to C and determines a polarity, fundamental in the geometry of the curve, in which each point of C corresponds to the osculating prime to the curve there; the cubic primal is the locus of chords of C. Consideration of the collineations of C into itself leaving an arbitrary point of space fixed leads to the study of a triply infinite system of lines g, one passing through each point of space. The osculating planes of C meet in pairs in the points of a surface K; this is a projected surface of Veronese and the lines g are its trisecants, and the curve C is asymptotic on K. The lines g are common to three linear complexes and include the tangents of C as special cases.

The geometry of the normal quartic is thus intimately bound up with that of many other familiar loci, concerning which there is an extensive literature. But the writer of the present tract has chosen, unfortunately as it seems to the reviewer, to develop the necessary properties of these loci *ad hoc*, always with a special eye on the normal quartic, and thus fails to relate the properties of the curve with the general scheme of four-dimensional geometry. To quote only one instance, the properties of the projected Veronese surface K are developed directly from its definition as the locus of points of intersection of pairs of osculating planes of C, which is clearly not the simplest way of initiating the geometry of this surface. To make things still more difficult for the inexpert reader, the references given in the text are of the scantiest, the author preferring to refer to the relevant articles in the *Encyklopädie* for details, and it is thus not clear which parts of the work are the original contributions of the author and which are derived from older sources.

In the latter part of the tract the rational quartic in ordinary space is considered as the projection of the normal curve from a point of its containing space. This method of approach enables the principal properties of the curve to be deduced in a simple manner, and brings out clearly the nature of the projectively special curves. The notion of apolarity is prominent throughout.

There are a few slips; the statement in the first example on page 7 is incorrect, and there is a curious misprint on page 40, but otherwise the pro-

duction of the work leaves little to be desired. As an account of the properties of the quartic curve it should prove of interest to synthetic geometers, but its value as a work of reference would be enhanced if it were more fully documented.

J. A. TODD.

Engineering Mathematics. By D. MCMULLIN and A. C. PARKINSON. Pp. viii, 266. 4s. 1936. (Cambridge)

The authors have aimed at presenting the groundwork of mathematics for the first year of a senior technical course so that it will enable the student to retain easy contact with the pure mathematics of his pre-technical course and yet contain a sufficiency of technical allusions to awaken interest without introducing engineering problems beyond the student's experience. This is a very sound point of view.

The fundamental parts of the beginning of algebra are well explained and, very wisely, a larger part of the book is devoted to elementary deductive geometry than is common in this course at present.

Unfortunately the book is marred by having only two pages on curved graphs, both of these being graphs of quadratic expressions, and by some statements liable to lead the student astray, such as " the curve is the graph of $x^2-2x-3=0$ ". With these exceptions it is very suitable for a mechanical engineering student, but not so suitable for the electrical student. There are very few examples taken from electrical engineering, and the graphs of the sine and cosine, which the electrical student should meet early in his course, are not mentioned.

The book is very attractively printed, has good figures and its price is within the reach of the technical student.

H. V. LOWRY.

Examples in Practical Mathematics. Second Year (Senior) Course for Technical Colleges. By L. TURNER. Pp. 96. 1s. 6d. 1936. (Arnold)

These examples, for students in technical colleges who are taking the second year of the national certificate course, are a sequel to the examples by Mr. Turner for the first year of the course. The examples are well chosen and are set out under thirty different headings, so that, roughly speaking, each exercise covers the work on one day of the course. Many of the examples are taken from the elementary parts of engineering theory, which the students will be studying about the same time; we are glad to find that there are few examples from difficult engineering problems, which are far beyond the student at this stage, and are no more practical to him than the problems one finds in the older algebra books.

In many colleges parts of this book are not dealt with till the third year of the course, but it is probably desirable that the range covered should be slightly beyond what most students are likely to need. At the same time we feel that the book would have been of greater value had it included a far larger number of easy examples for revision of the first year work. Apart from this criticism, we are sure that both teachers and students will find these collections of examples very useful.

H. V. LOWRY.

The Marks of Examiners. By SIR PHILIP HARTOG and E. C. RHODES: with a Memorandum by CYRIL BURT. Pp. xix, 344. 8s. 6d. 1936. (Macmillan)

The first part of this book contains a detailed account of the marking, under the auspices of the International Institute Examinations Enquiry, of scripts in various examinations. A summary of the results, and of the conclusions which Sir Philip Hartog and his collaborator drew from them, was published

recently as a separate pamphlet, *An Examination of Examinations*, which attracted a good deal of attention at the time, and was noticed in the May issue of the *Gazette* by the present reviewer.

The preliminary account is followed by a section for which Dr. Rhodes is alone responsible. A brief discussion of differences of standards between various examiners, differences due to random variations, and precision of marking, leads him to the question of ideal marks; and on the assumption that for every candidate t there exists a set of equations of the form $X_t = Q_t + A_t$ (X_t being the mark allotted by examiner A, Q_t the ideal, A_t the examiner's personal difference), Dr. Rhodes develops a method of calculating the ideal mark. The analysis is then applied in considerable detail to the marks tabulated earlier in the book. In a later section Dr. Rhodes investigates the consequences of assuming $X_t = r_a Q_t + A_t$, r_a being a multiplier peculiar to the examiner A, and thus admitting the possibility of each examiner having a different set of ideal marks, some spreading them more than others. This second approximation gives results which do not appear to differ widely, when applied to the present investigation, from those obtained by the first method.

Professor Burt then contributes a long and valuable memorandum in which the problem of analysing examination marks is approached from a different angle. He suggests that an examiner's actual marks depend on six factors: the standard of severity; the distribution of the marks; the true value of the candidate's work (it is a relief to find that he does not overlook the possibility that this may have *some* effect upon the marks allotted); limited influences—such as might bias two or more of a panel of examiners; personal influences, peculiar to an individual examiner; accidental influences (*e.g.* the examiner goes to sleep in the middle of his marking). He thus sets himself a problem of such generality as to be insoluble; so, having marched his soldiers up the hill, he now marches them down again, and points out that in practice there are several methods of simplifying the analysis without undue loss of accuracy. The resulting development, too long to be summarized here, is of considerable interest and repays careful study. Whether it could be applied without further simplification under the conditions of the modern large-scale examination is a little doubtful.

Dr. Burt, though generally in agreement with Dr. Rhodes, considers that some of his own formulae are more reliable than those of his collaborator. Their two methods give results which are in substantial agreement, but they might not do so, Dr. Burt thinks, if Dr. Rhodes had been testing larger groups. If a criticism may be ventured, it may be that the sizes of the groups tested invalidate more conclusions than either contributor thinks. For example, Dr. Rhodes says (p. 243): "The general idea that mathematics and science subjects can be marked with greater precision than humanistic subjects is apparently not founded on a sound basis". This conclusion itself is apparently not founded on a sound basis, for it appears to be based, so far as mathematics is concerned, upon an examination of twenty-three (!) Honours scripts. Ten thousand School Certificate scripts is the foundation the reviewer would have preferred. But possibly on this point the doctors disagree, for Dr. Burt, in a footnote on p. 292, in another context, takes "the accuracy of the mathematicians" as his ideal.

The book closes with a short memorandum on certain points of difficulty in connection with School Certificate examinations and replies to some criticisms of *An Examination of Examinations*.

The book is less sensational than the earlier summary and is likely to be correspondingly more valuable. The tables are interesting and much of the analysis is suggestive. Most of the conclusions, to examining bodies at least

who have been studying these problems for many years, are not very startling; and if they tend to be either obvious or, in a few cases, dubious, at least the book has provided some amount of statistical backing for opinions previously held empirically.

Many examiners have vague impressions of the following kinds: it is harder to mark problems in mathematics than computations; manipulations of marks in an office can iron out differences of standard between various examiners but not random variations; using examiners in pairs is successful in reducing random variations; in any examination in which marks for different subjects are added, much harm may be done if the spreading of the marks differs widely in the various subjects; and so on. Such examiners will probably find their ideas becoming clearer and taking more precise form after reading this book.

But though the book may do good in a limited field, it must be remembered that it deals with one aspect only of examining, and that not the aspect which, in the reviewer's opinion, is the most in need of investigation; and the book ought not to distract attention from problems of an altogether broader kind which the remarkable growth of the examining system in this country has brought to the forefront. B. A. H.

Differential Geometry. By W. C. Graustein. Pp. xii, 230. 1935. (Macmillan, New York)

I find it hard to convey in a short notice my impression of this account of the classical theory of curves and curved surfaces. There are many details on which I feel argumentative, but the book is one to be welcomed. There are no new ideas, nor are the old ideas recast: this is perhaps fortunate, for the copyright notice warns the reader against disseminating any knowledge he may acquire. If I say that the book is a useful addition to the learner's library, I hope I am paying it the tribute which the writer desires.

Let me give two examples of details which I should like to see altered. First, there is no sense in which $a_1 + a_2$ can properly be called a mean of a_1 and a_2, and the use of language according to which * the mean curvature of a sphere of unit radius is 2 is as unfortunate as it is common; no one who consults Sophie Germain's own paper can fail to see that the case for her original and more logical meaning of "mean curvature" is overwhelming. Secondly, conjugate tangents are introduced and defined as conjugate diameters of the Dupin indicatrix; the attention paid to conjugate tangents would be quite inexplicable if this was the real genesis of the idea. Of course, Dr. Graustein proceeds at once to establish as a theorem their organic property in relation to the surface, but a learner does not easily recover from a false emphasis, and the symmetry of the relation between conjugate tangents instead of being recognized as a fundamental theorem in the theory of surfaces is taken over without comment or appreciation.

For the complementary picture, broad outlines are better than a few details Since kinematical language is avoided systematically, the book is no more to be condemned on that account than for not being a treatise on elliptic functions or on any other subject which the author is not attempting to expound. It is written in terms of vectors. In the preface we read: "I have employed Study's notation. . . . There are other notations which are entirely adequate. If the reader is already familiar with one of them, I can assure him from personal experience that he will have no difficulty in translating the notation of Study into his own, and that his grasp of the algebra of vectors will undoubtedly be strengthened by the process". From a man who writes in this

* This way of exposing the absurdity I owe to Dr. R. Rado.

way on a matter which usually excites the most savage and theological passions, we can be sure of a well-balanced treatment of any subject.

In some respects the range is limited surprisingly: geodesic torsion is not related to normal curvature, and integral curvature is evaluated only for a geodesic triangle. On the other hand, the concept of the absolute geometry of a surface is presented very clearly, and Levi-Cività's parallelism is explained. Nowhere is the analysis allowed to outgrow its purpose, and the book is a careful and stimulating introduction to a subject which remains in the author's hands a fascinating branch of geometry. E. H. N.

Vom Punkt zur vierten Dimension. Geometrie für jedermann. By E. Colerus. Pp. 445. 1935. (Zsolnay, Berlin)

Unlike its predecessor, *Vom Einmaleins zum Integral*, this book will not enable the reader to retrieve lost opportunities of a misspent youth. The logical and philosophical problems of analysis lie deep and out of sight, and anyone may learn to solve equations and to calculate derivatives without misgivings regarding the concept of number. But if our critical faculties are developed even slightly when we approach geometry, we are puzzled at the outset by the nature of congruence and parallelism, and by the contrast between the approximative character of measurement and the precision of geometrical theorems. We appreciate readily that the basis of geometry must be axiomatic, but this does not explain the relation between our experience of the external world and our geometrical intuition.

Herr Colerus guides us from point to fourth dimension along a philosophical track that is far removed from the well-worn highway of school geometry. He does not show us geometry as Klein saw it, a matter of transformations and invariant properties, nor is he instructing us in the technical processes which it is the mathematician's delight to master. Although many elementary theorems in pure geometry, in plane and spherical trigonometry, and in analytical geometry, are proved, the teacher will find his profit not in hints for classroom use, but in the intense satisfaction of facing questions which are usually shirked, and in the resulting power to meet the "Why?" of the most intelligent of his pupils as confidently as the "How?" of the stupidest.
E. H. N.

Premières Leçons sur la Théorie Générale des Groupes et ses applications à l'arithmétique, à l'algèbre, à la géométrie. By G. Bouligand. Pp. vi, 242. 40 fr. 1935. (Vuibert)

Once again Professor Bouligand has shown that he can wear the mantle of the Klein who was president of the International Commission on the Teaching of Mathematics and author of *Elementarmathematik vom höheren Standpunkte aus*. His purpose in this book is to make the theory of groups play its part in a general mathematical education. His approach is individual and arresting:

Un groupe est un domaine de causalité, vu que des hypothèses (ou causes) invariantes par les modifications du groupe engendrent des conclusions (ou effets) qui se conservent aussi par ces modifications. . . . A chaque proposition mathématique est désormais associé un groupe, celui de toutes les modifications menant d'un cas d'exactitude à un nouveau cas d'exactitude. Ce groupe sera le domaine de causalité de la proposition.

Illustrated by simple examples this is convincing. We cannot question the importance or universality of the concept, and we are impatient to learn how it is to be transferred from the verbiage of philosophy to the technique of mathematics.

Five chapters are occupied with first principles. The headings of the last six are as follows: VI—Les groupes définis au moyen d'un nombre limité

d'opérations ; VII—Les transformations infinitésimales et la génération des groupes continus. Applications aux équations différentielles ; VIII—Les groupes et l'arithmétique ; IX—Les groupes et la résolution des équations algébriques ; X—Les groupes et la géométrie ; XI—Les groupes et la formation des algorithmes.

Among the sectional headings are : in Ch. VIII, Le groupe de classes attaché aux résidus quadratiques ; in Ch. IX, Esquisse de la théorie de Galois ; Groupe d'une équation algébrique ; in Ch. X, Figuration des éléments imaginaires d'après divers modes ; Indications sur les géométries non holonomes ; in Ch. XI, Le calcul extensif de Grassmann ; Application de la multiplication extérieure au calcul intégral.

To anyone who protests that this is a lazy review, I can only reply that I have worked hard and vainly to substitute a better account of the book in my own words. And to anyone who says that over such a range the treatment must be worthlessly scrappy, I can only reply " Read Prof. Bouligand's book and confess that you are wrong ". E. H. N.

The Teaching of Arithmetic through Four Hundred Years (1535-1935). By FLORENCE A. YELDHAM. Pp. 143. 5s. 1936. (Harrap)

This book is mainly a study of arithmetical textbooks from 1535 to the present time, with illustrations from famous arithmetics such as those of Recorde and Cocker. In the first chapter the author passes under review their aims of teaching the subject. In Chapter II, which is fairly typical, she discusses the *De Arte Supputandi*, 1522, of Cuthbert Tonstall, who performed subtraction by the method of equal additions, but whose method of setting out division has not survived. He included series (simple arithmetical) in his first book and postponed fractions to Book II. He dealt with Proportion, solved by " means product÷extreme ", in his third book, and, unlike some later writers, avoided the mistake of comparing unlike terms. In Book IV he treated series and proportion, and the Rules of False Position which held their own until the use of the algebraical solution in the nineteenth century, as Miss Yeldham shows by comparing his work with that of later writers. By following this plan of description and comparison throughout the book the reader is shown the history of the methods of teaching specific topics of arithmetic, with variations and aberrations (*vide* Ch. VII on the use of verse) of the subject. The historical survey is helped by an excellent index. It is a pity that the author bases her discussion of Recorde's *Arithmetic* on the 1668 edition when copies of one or two editions of the previous century are accessible.

Miss Yeldham has carried out her plan well and produced an interesting book which can be recommended to teachers of arithmetic, particularly intending teachers, whose interest in special method would be stimulated by this survey of its historical development. The book should also find a place in the secondary and central school libraries, though its title might deter some pupils.

I am not quite happy about the title. Is it justifiable to infer from textbooks alone how the subject was taught ? Perhaps that is all we can do for an early period. Schoolmaster John Brinsley, in the *Grammar Schoole*, 1662, advised " seeke Records Arithmetique, or other like Authors and set them to the Cyphering Schoole " for those who wanted to do more than merely write numbers readily. For the nineteenth century there are reports of inspections and enquiries, and histories of schools, which not only throw light on teaching methods but on some of the broader questions connected with the teaching of arithmetic. Miss Yeldham has nothing to say about the tech-

nical application of arithmetic, apart from commercial, which became important with the development of industrialism early in the nineteenth century. This aspect might have been suggested by the work of L. and T. Digges in the sixteenth century, which, however, she does not mention. Besides technical arithmetic we may mention the appearance of mental arithmetic as a separate subject, and particularly the teaching of Pestalozzian method in the training colleges. Then there are questions such as the status of arithmetic and the teacher, and the extent to which the subject was taught in public schools and others. She does discuss the nature of arithmetic, bringing out how pure arithmetic, since the Renaissance, has been largely supplanted by applied, and referring to certain modern developments, *e.g* arithmetic of citizenship. But nevertheless she must be held guilty, as far as the nineteenth century is concerned, of making bricks without straw in attempting to write an account of " the teaching of " without reference to any literature dealing specifically with " the teaching of ".

Let not this criticism, however, detract from the value already placed on the book. An adequate discussion on the teaching of arithmetic through four hundred years would need more than about 150 pages. In a small space Miss Yeldham has covered a great deal and produced a very readable book. It is an excellent introduction to the subject for a serious student of the history of the teaching of arithmetic. R. S. W

Hints to Travellers. I. Survey and Field Astronomy. By E. A. Reeves. Eleventh edition. Pp. viii, 448. 16s. 1935. (Royal Geographical Society)

Hints to Travellers was the title of a small pamphlet prepared in 1854 by the Royal Geographical Society's Instructor in Surveying. The eleventh edition is a substantial work containing the most up-to-date and authoritative information on Survey and Field Astronomy. The second volume, rewritten by various authorities on Equipment, Transport, Photography, Collecting and Health, will be published shortly.

The present volume is a wonderfully compact summary of all that a traveller might wish to know about instruments and the work for which they are used. It contains first a very careful and well-illustrated account of the instruments themselves, with clear directions for adjusting and using them. The instruments dealt with include the $3\frac{1}{2}$-inch micrometer theodolite, which has now superseded the 5- or 6-inch instrument for exploratory survey, and the prismatic astrolabe, an instrument for observing the time at which a star reaches a fixed altitude, 60° or 45°.

This chapter is followed by a summary of trigonometrical formulae, with notes on map scales and projections. The next chapter, on " Geographical Surveying and Mapping ", deals fully with a variety of methods, from the most accurate triangulation and levelling, down to rough work with prismatic compass and aneroid. It includes interesting sections on reconnaissance by motor-car and on photographic surveying. In this, as in other chapters, directions are usually given without explanation. Of the exceptions to this rule, the explanation, on p. 119, of the second method for orientating a plane-table seems unconvincing.

In the chapter on Field Astronomy there is first an account of the older methods of determining latitude and longitude, with a special section on wireless time-signals. There is also a full treatment of position-line methods, now increasingly used on land as well as at sea. The methods described vary from the simplest possible, using altitude and azimuth tables, to the most accurate developments, based on observations with the prismatic astrolabe.

At the end of the book there are two star charts and a considerable col-

lection of tables, including logarithms of 4-figure numbers, given to 6 figures, with differences and proportional parts, and logarithms of the trigonometrical ratios, at intervals of 30 seconds of arc, given to 6 places, with proportional parts. E. H. L.

Mathematics and the Question of Cosmic Mind, with Other Essays. By C. J. Keyser. Pp. v, 121. 75 cents. 1935. *Scripta Mathematica* Library, 2. (*Scripta Mathematica*, New York)

This interesting collection of essays will serve, no doubt, to complete the picture of the philosophical attitude of Professor Keyser, who has given us so many means of appreciating the philosophical aspect of science in general and of mathematics in particular. The two essays on The Meaning of Mathematics and The Bearings of Mathematics are illuminating and useful to the teacher of mathematics. In discussing the question of the Cosmic Mind, Professor Keyser admits that science and mathematics have revealed the universe to be not a chaos but a cosmos, a veritable intelligible world, so that he can say " with almost perfect confidence " that the universe is essentially and ultimately a Realm of Mind. T. G.

Portraits of Eminent Mathematicians. Portfolio I. With brief biographical sketches by D. E. Smith. $3.00. 1936. (*Scripta Mathematica*, Amsterdam Avenue and 186th Street, New York)

This collection is warmly recommended to the many teachers who feel that such portraits help to stimulate the interest of their pupils. The plates are beautifully engraved, being large and clear. Since the selection from available portraits has been made by Professor Smith from his own fine collection, there is no need to praise it, while the brief biographies should appeal to the young student. Several facsimiles of handwriting are given.

The twelve mathematicians dealt with in this first portfolio (we hope there will be others to follow) are Archimedes, Copernicus, Viète, Galileo, Napier, Descartes, Newton, Leibniz, Lagrange, Gauss, Lobachevsky, and Sylvester. The Archimedes plate is a reproduction of a mosaic, depicting the death of Archimedes, found at Pompeii. The rest are from life portraits. T. A. A. B.

Histoire des Sciences. Antiquité. Pp. 1224. 200 fr. 1935. (Payot, Paris)

This excellent work, which is perhaps the most comprehensive compendium of ancient science, especially during the classical period, is of special importance to those interested in the teaching of the mathematical and physical sciences, as it provides them with the historical background of their subject. The value of the history of science becomes every day more and more acknowledged as an almost indispensable complement of the study of any science. And this is particularly true of the mathematical and physical sciences in so far as many problems which are classical have been invented and solved in classical antiquity by the founders of our Western civilization.

Mathematicians will be particularly grateful to M. Mieli and M. Brunet for the care and thoroughness with which they have treated the important questions dealing with the mathematical methods and discoveries of the Greeks. After a short survey of the early theories connected with the Ionian and the Pythagorean schools, and a fuller sketch of the methods familiar to the Platonist school (with a chapter dealing with the three famous problems of the quadrature of the circle, the duplication of the cube and the trisection of the angle), the authors deal at length with the golden period of Greek science—that is, with the discoveries and methods of the giants of the Hel-

lenistic or Alexandrine period. This portion of the work, which is illustrated by translations from the authors considered, is one which should be meditated by all mathematicians.

In the field of pure mathematics, Euclid, Archimedes and Apollonius are treated with the prominence they deserve. The study of the works of Archimedes is far from being complete; and the authors believe that much could be done by using the available Arab manuscripts. In a lengthy note on p. 38 they explain the meaning of the famous "Method" of Archimedes as involving the notion of "integral", and of "Static moments" considered as proportions and not as products of a distance by a force. It is thus suggested that this method, which was known in the Middle Ages, has influenced the conceptions of Cavalieri and the genesis of the Calculus.

The astronomical views of Heraclides and Aristarchus, and their bearing on the heliocentric view of the universe, are summarized and contrasted with the doctrine of the homocentric spheres developed by Eudoxus. And so are the remarkable labours of Hipparchus, who improved the theory of the epicycleased trigonometrical methods in astronomy, and of Eratosthenes, the founder of scientific geography. A good account is also given of Ctesibius, Philo and Heron, the most representative engineers of classical antiquity. Of the later astronomers, Ptolemy is given fifty pages in which a summary of his works is followed by a helpful appreciation and criticism from the authors. An interesting addition to the traditional exposition of Ptolemy are the two chapters on optics and acoustics, in which an adequate account is given of the methods and views of the ancients concerning these important subjects. The exposition of Greek mathematics closes with an analysis of the writings of Nicomachus, Diophantus, Pappus and Proclus.

The various historical and scientific problems raised in this excellent history are carefully explained in the light of available texts, while conflicting opinions about them are fully discussed in a wealth of footnotes. The views expressed are illustrated and supported by nearly three hundred textual quotations from more than eighty original sources. A number of passages have been translated by the authors themselves. A comprehensive index, two synchronical tables and a full critical bibliography add to the practical and scholarly value of this magnificent work.

THOMAS GREENWOOD.

Esquisse du Progrès de la Pensée mathématique. By J. PELSENEER. Pp. 160. 18 fr. 1935. Bibliothèque Scientifique Belge, 23. (Thome, Liége)

This sketch of the current of mathematical ideas makes interesting and informative reading; I hope that M. Pelseneer will not remain content with a sketch, but will sometime give us part at least of a finished picture. The present work is good evidence of his ability to do so.

The author's task, suggested by Boutroux' valuable *L'Idéal scientifique,* is to describe climates of mathematical thought—the slowly-gathering electrical conditions from which eventually the revealing lightning flash emerges. There are five sections: the primitives; the Egyptians, Sumerians and Babylonians; the Greeks; the age of Descartes; the nineteenth and twentieth centuries. In the first section the author illustrates the logical and mystical ideas about number, and comments on the absence of geometry, by observations drawn largely from the primitive races of the Belgian Congo. The later periods have their own documentary evidence till in dealing with the moderns every phase is illustrated by apt quotation.

Naturally M. Pelseneer expects his readers to have a reasonable acquaintance with the ordinary chronological history of the subject. Given this knowledge, the book is easy reading. Among its merits is the fact that almost every

sentence will cause the cautious reader to think and to ask questions, some of which his own knowledge and industry will allow him to resolve. But on many points a lengthier exposition would be welcome. To take one example at random, in speaking of Newton M. Pelseneer says: " Pour lui, l'analyse infinitésimale n'a jamais été qu'un expédient, et d'ailleurs, — la place nous fait malheureusement défaut pour insister sur ce point — elle n'introduisait aucun idée nouvelle opposée à celles qui étaient à la base de l'algèbre finie ". The question cannot be discussed here, but it would be in place in that larger and more detailed work which I hope M. Pelseneer will write. T. A. A. B.

Exposés de Géométrie Cinématique. I : Cinématique du solide et théorie des vecteurs. II : La masse en cinématique et théorie des tenseurs du second ordre. III : Cinématique des milieux continus. By C. PLATRIER. Pp. 55, 83, 35. 12 fr., 18 fr., 8 fr. 1936. Actualités Scientifiques et Industrielles, Nos. 325, 326, 327. (Hermann, Paris)

The pages and the chapters in these booklets are numbered independently, but the paragraphs are numbered in one succession, and the booklets form a connected course on the elements of the kinematics of a material system. The purpose is to give the student of applied mathematics and mathematical physics such geometrical preparation that his attention need not be diverted from mechanical and physical principles. To some extent this is a common, indeed an inevitable, practice: analysis of screw motion must precede the dynamics of a rigid body without a fixed point, and we cannot write down hydrodynamical equations in any form until we have considered how motion in a continuous medium can be specified. But nobody learns the kinematics of pliable or fluid bodies before proceeding from the kinematics of a point to the dynamics of a particle, and it is to the logician rather than the teacher that the consecutive abstraction of the subject is interesting.

The average teacher insists less than the logician, though there is no reason why he should, on knowing in what sense there are kinematical theorems expressible in terms of relative motion alone, that is, kinematical theorems independent of the assumption that some particular frame of reference is at rest. This investigation, on which M. Platrier lays proper stress, is all the better for being conducted in an atmosphere even more rarefied than that in which projectiles describe parabolas and pendulums oscillate for ever, and although it is hard to see a place in the English teacher's working library for M. Platrier's course as a whole, sections of it form essentially a better approach to the theory of relativity than accounts of the real experiment of Michelson and Morley or of imaginary experiments with moving rods and clocks.

E. H. N.

Soviet Science. By J. G. CROWTHER. Pp. x, 342. 12s. 6d. 1936. (Kegan, Paul)

It would be entirely proper on occasion to comment in these pages on the sociological significance of a work that was primarily mathematical, but when the work itself is not mathematical the *Gazette* has no business with it except to discover its mathematical interest. If this principle is sound, I must not speak of the mixture of philosophy, anecdote, and exposition which constitutes Mr. Crowther's account of the great laboratories and scientific institutions of Russia and of the problems which are being attacked in them. The book is first-hand information, the outcome of frequent and lengthy visits; Mr. Crowther knows the language of the country, and is a trained scientist, an experienced observer, and a practised writer.

The book is in six sections: theory and organization, physics, chemistry, applied science, biology, the history of science. It is composed for the general

reader, and although the technical vocabulary is extensive, no symbols are used. The result is that Mr. Crowther cannot extend to mathematics his ability to convey the essence of a problem in pure or applied science, and a few paragraphs on the theory of non-linear vibrations which Liapounov and Mandelstamm are doing so much to develop contain his only serious attempt of the kind. By way of compensation it is to the mathematician that the sixth section, which is very short, particularly appeals, for the names to which most attention is given there are those of Newton and Lobachevsky.

As Professor Levy and others have made clear to the English reader, communist philosophy requires that while the creative genius may emerge in any society, the product of his activity even in the most abstract fields of thought is conditioned by the social structure of which he is part. The retort that the law of gravitation has nothing to do with politics or economics is as obvious as the invitation to the idealist philosopher to kick the stone whose reality he is disputing, and the Soviet historians of science accept the challenge. It is difficult for us in England to perceive that with the same mathematical equations as Newton's a very different picture of the universe might well have been associated, but Hessen and his colleagues are prepared to show us how much the *Principia* owed to the community. We are told that they have in preparation a new collected edition of Newton's works ; they can hardly have secured access to the material which has been thought essential here to a definitive edition, but unless their bibliographical ideas are very different from ours they must have overcome some of the difficulties that have been pronounced insuperable hitherto, and the outcome of their enterprise must be awaited eagerly.

Nothing could illustrate the dependence of the culture of Eastern Europe on that of the West better than the local estimate of Lobachevsky. As long as it was unrecognized in France and Germany, his great work was regarded in Kazan as the regrettable but harmless eccentricity of a man who in other respects was sound, and whose administrative powers earned him the highest academic honours in spite of it. In the belated appreciation of his genius, Kazan merely fell in, however readily, behind the rest of the civilized world. Here we find no political moral, and as mathematicians we are grateful to Mr. Crowther for his portrait of the popular, successful, unembittered controller of the university where, we may be sure, the mention of the theory of parallels always evoked a kindly smile. E. H. N.

A Modern Elementary Trigonometry. By W. S. Catto and F. J. H. Williams. Pp. 263. 3s. 6d. 1936. (Harrap)

The primary purpose of this book is to cover the work in trigonometry for the higher grade papers of the Scottish Leaving Certificate. The authors are members of the mathematical staff of George Watson's College for Boys, and they have written—for the most part—as teachers of mathematics rather than as mathematicians.

Unfortunately they are the latter when dealing with the proofs of $\cos(A \pm B)$ and $\sin(A \pm B)$, which are done by means of projections. Now I do not believe that any pupil of only average ability can be taught these proofs by projection, although Messrs. Catto and Williams explain them as well as one could expect, but they are very careful in their diagrams to keep A, B and $(A+B)$ as acute angles. Naturally the proofs are valid for all values, but the authors seem to recognize the intellectual limitations of their pupils in this way. Therefore, I do not agree with the inclusion of these proofs, firstly, because from experience I find that they are not for the average pupil. Secondly, since 1925, when the higher grade papers in the examination were

reduced from three in number to two, proofs of the formulae $\cos(A \pm B)$ and $\sin(A \pm B)$ have been asked several times, but only for acute angles. It is extremely unlikely that proofs for all values of A and B will be asked. Thirdly, easier proofs for all values may be done by means of the first formula in the coordinate geometry of the straight line, viz. $d^2=(x_1-x_2)^2+(y_1-y_2)^2$, a formula which must be known by the pupils for whom the book is primarily intended. (See Durell and Wright, *Elementary Trigonometry*, p. 209.) Incidentally, as the examination is becoming more analytical in character, as the new "Note" from the Scottish Education Department will clearly indicate, an easy analytical proof seems preferable to a difficult geometrical proof.

The remainder of the book is very good. In each chapter several examples are worked in full in the text, and there are a great many to be attempted by the pupil. Important formulae are given in heavy black type, and a list of them is given at the beginning of the book. The authors recognize that pupils can memorize formulae without knowing anything about the meaning or application of them. Consequently, having proved a formula, *e.g.* $\sin^2\theta+\cos^2\theta=1$, they then give it again with three or four different values of θ, *e.g.*

$$\sin^2 3A+\cos^2 3A=1, \quad \sin^2 \tfrac{1}{2}A+\cos^2 \tfrac{1}{2}A=1, \quad \sin^2 32^\circ+\cos^2 32^\circ=1.$$

This is done for all the formulae that are not connected with the triangle; it is a point that is frequently overlooked by teachers, and consequently a pupil who knows that $\sin^2\theta+\cos^2\theta=1$ looks aghast when asked the value of $\sin^2\dfrac{39A}{2}+\cos^2\dfrac{39A}{2}$. The geometrico-trigonometrical formulae connected with the triangle are proved in full, viz. when a given angle is (i) acute, (ii) obtuse, (iii) right, thus fulfilling the requirements of the examination. The ambiguous case is treated clearly, *i.e.* with illustrative diagrams of each possibility. It is a pity that all the examples given on "Heights and Distances" are numerical, as the last question in Paper I is usually a "literal" problem, frequently taken from Hobson! In the chapter on Equations five different types are explained, at least two examples of each type being worked in full. Graphs are adequately treated and Radians are in their proper place—at the end of the book.

In short, with the exception already mentioned, the book is to be recommended for fourth and fifth year classes in Scottish secondary schools as it is quite obviously the work of two expert teachers. ALEX. INGLIS.

Revision Test Papers in School Certificate Mathematics. By W. A. EVANS and J. W. HARDING. Pp. 56. 1s. 6d. 1936. (Macmillan)

This very good collection of examination questions in Arithmetic, Algebra, Geometry and Numerical Trigonometry should be of great service to pupils sitting the School Certificate or the Scottish Leaving Certificate. There are ten test papers in Arithmetic with nine examples in each; at least four or five in each set are straightforward questions testing accuracy. The Arithmetic section is the best of the four. The ten Algebra papers (eight questions in each) are very good, but more examples on Ratio and Proportion, Variation, Fractional and Negative Indices would have improved the collection. If the four sections were put in order of merit the Geometry section would appear second. There are ten papers with nine questions in each, many of which are of the "theorem + rider" type, and most of which are very good indeed. Numerical Trigonometry is represented by five papers (four questions in each), and these are quite good. Any pupil who has worked intelligently through the 180 questions in this book ought to be quite "safe". The price is rather high for a book of this kind, for after all it will be an "extra" in the mathematical equipment. ALEX. INGLIS.

Höhere mathematik für mathematiker, physiker und ingenieure. IV, 3. By R. ROTHE. Pp. 49. RM. 1.50. 1936. Teubners mathematische Leitfäden, 35. (Teubner)

Dr. Rothe's admirable book is nearing completion; the fourth and final volume contains examples with full solutions or hints, and the present part of that volume is concerned with the integral calculus. The examples are carefully chosen and cover a very wide field, so that the pure mathematician, the physicist, the chemist and the engineer may all find matter to their taste.

T. A. A. B.

Tables of Physical and Chemical Constants and some Mathematical Functions. By G. W. C. KAYE and T. H. LABY. 8th edition. Pp. 162. 14s. 1936. (Longmans)

The first edition of these tables appeared in 1911, the compilers declaring themselves impressed by " the need for a set of up-to-date English physical and chemical tables of convenient size and moderate price ". Eight editions in twenty-five years is clear evidence of the ability with which that need has been met. Naturally the new edition has been brought up-to-date, but no extensive additions have been made. T. A. A. B.

A School Geometry. By C. W. GODFREY and R. C. B. TAIT. Pp. viii, 203. 3s. 1936. (Blackie)

A School Certificate Geometry. By W. G. BORCHARDT. Pp. vii, 324, xxii. 4s. In two parts, 2s. 6d. each. 1936. (Rivingtons)

These two books illustrate the prevailing tendency of authors of textbooks in elementary mathematics to write, not, ostensibly, because they feel that they have some definite, original, ideas as to how the particular subject should be taught, but to provide a course fitted to the syllabus of some public examination or examinations.

The preface of the first of these starts off with the sentence " This text-book is intended to provide a course in Geometry leading up to the School Certificate ", and goes on to explain that an appendix renders the book suitable for other examinations as well. The book strikes a very impersonal note. It is scarcely a textbook in the accepted sense of the word, for, as the authors explain, " the usual ' talk ' has been omitted ". The text consists merely of such definitions, proofs of standard theorems, and constructions, as are required for examinations. This is followed by nearly 80 pages of exercises taken, for the most part, from papers set in certain public examinations.

The book is printed in exceptionally clear type and has a pleasant, uncrowded, appearance. The figures are well drawn, with all construction lines thin.

There are very few points that will call for notice or criticism. The three standard congruency theorems are enunciated, but the proofs are left blank in the main portion of the book and relegated to the appendix. The reason for this relegation is that certain examining bodies do not require candidates to reproduce proofs which depend on the method of superposition, because this method is regarded as logically open to suspicion. But when the authors come to set out the proof of the right-angle—hypotenuse—side case of congruency, they quite unnecessarily express it in terms of superposition. The statement " the lines joining the ends of equal and parallel straight lines are themselves equal and parallel " requires qualification.

For those teachers who rely on oral instruction and require for their pupils a book which contains the bare bones of geometry with some well-selected and well-arranged exercises, this should serve the purpose well.

Mr. Borchardt's book is more personal. It is not merely a " School Certificate " geometry—in fact the titles of these two books might aptly have been exchanged. The book follows the usual course of a first stage in which the stress is laid on the discovery of fundamental facts by common-sense methods based on carefully drawn figures, and a second, exclusively deductive, stage. The text (what the authors of the first book refer to as the " talk ") is lively, clearly expressed, and individual. There are many interesting digressions, some of them on solid geometry. Like the other book, the printing is good, the pages not overcrowded, and the figures clearly drawn. There is a profusion of well-graded exercises. In general, the examples are arranged in two sets, the first containing numerical and constructional exercises on a theorem or group of theorems, the second a set of " riders " on the same. There are 80 revision papers placed at intervals throughout the book.

The reading of these books has provoked in the writer's mind certain reflections on points connected with the teaching of elementary geometry. These are set out here; though they are associated with and illustrated by the texts of the two books, they are not intended to be serious criticisms which would invalidate the general impressions left by them.

(i) The difficulty of writing the early, mostly inductive, part of a geometry book is that one is bound to use terms which lack precision. Purely geometrical terms such as those which are used as the basis of a deductive course can be exactly defined. But what exactly is meant by the word " direction " ? Many authors trip up here. Mr. Borchardt is for the most part consistent; but in Exercise 6, No. 3, he asks, " Are all horizontal lines in the same direction ? " (the answer being, presumably, " No "), whereas in No. 3 the question begins, " What is the direction, vertical or horizontal, of . . . ".

There is a regrettable use of the word " deduce " on p. 29. After some general arguments, based on " direction", the three tests for parallelism are stated, in the form of a theorem. Then follows the statement, " If the two straight lines are parallel, then we *deduce* the results of the next theorem " (which is, of course, the converse, the great stumbling-block of any logical sequence). The use of the word " deduce " here confuses issues both as regards the meaning of deduction and of converses.

(ii) Why are books on elementary geometry usually so uninformative on the subject of converses ? Why is it that the student is required to prove a number of converse theorems without his attention ever being called to theorems of which there are no converses, or geometrical statements which can be so worded that the converses are not necessarily true ?

(iii) Both these books are careful in giving exact references whenever the theorems on congruent triangles are applied. Both are careful to insert the word " corresponding " (so often omitted) in the case of " two angles and corresponding side ". But Mr. Borchardt slips up twice. On p. 61 " corresponding " is omitted, and on p. 121 no reason is given for the congruency of two triangles. A modern practice, not adopted in either of these books, of using initials (S, S, S; S, A, S; A, A, corr. S; R, H, S) has much to recommend it.

(iv) In any treatment of angles greater than two right angles the language needs to be carefully guarded. A careful reader would find his ideas in something of a tangle by the end of p. 10 of Mr. Borchardt's book. Although he expresses himself in simpler terms than these, the author defines the angle AOB as obtained by clockwise rotation of the radius vector OA into the

position OB. One of his figures shows AOB (Fig. 1) as an angle of three right angles.

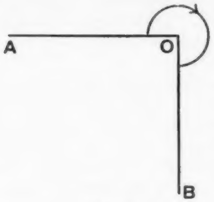


Fig. 1.

Without, therefore, explicitly saying so, he has stressed the idea of "sense" of rotation. But at the bottom of the page he gives the figure shown in Fig. 2

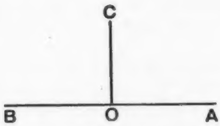


Fig. 2.

with the statement that $\angle AOC = \angle BOC$, each being a right angle. By the preceding, AOC is three right angles. $\angle BOC = \angle COA$ would have been consistent with what had gone before.

(v) Scarcely any books on elementary geometry or trigonometry which include a paragraph on the compass avoid statements which are obsolete or misleading. We find on p. 19 of Mr. Borchardt's book the usual implication that S. 30° W. and 210° mean precisely the same thing, and the paragraph ends with the singularly uninformative statement, "In military surveys, bearings are measured from the geographical North".

The following describes the current practice. In the Royal Navy and the Mercantile Marine the magnetic compass is still in partial use. Points of the compass are comparatively little employed. Courses and bearings relative to the magnetic North are given from 0° to 90° East and West of the North-South line. When these are corrected for magnetic variation, so as to refer to the geographical North, they are given as from 0° to 360°, clockwise from the North. Thus in current naval practice S. 30° W. and 210° do not mean the same thing. The first is reckoned relative to the magnetic North, the second relative to the geographical North. All the larger ships of the Navy are now fitted with the gyro-compass. The needle of this points to the geographical North. "True" courses and bearings can, therefore, be used directly and exclusively in such ships. In the Royal Navy the magnetic (or mariner's) compass is thus becoming of secondary importance. In the Army, bearings are taken by means of the prismatic (or other form of magnetic) compass. These have to be corrected for compass error (which may be considerable), and the further correction for magnetic variation will convert them to true bearings.

But in countries in which the grid system of maps is in use bearings are transmitted on "grid" bearings—grid North being within a few degrees of true (*i.e.* geographical) North. Bearings are never transmitted as magnetic.

They are always transmitted by the three-figure method, with the words "true" or "grid" following the figures; thus: 025 true, 312 grid.

(vi) It is interesting to compare the methods of proof in the two books under review of the theorem, "If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic".

In *School Certificate Geometry* we have: "Given $\angle ACB = \angle ADB$, to show that A, C, D, B are concyclic. Draw the circle through A, B, C and suppose that it does not pass through D, then D must be either inside or outside the circle. Let the circle meet AD or AD produced at K. . . ."

This is all very well, but it is logically incomplete. The circle through ABC need not meet "AD or AD produced". AD might be a tangent to the circle, or D might be so situated that DA produced meets the circle. It is surely better to defer this theorem till after the definition of tangent, and then to make the proof complete.

In *A School Geometry* we have the proof (found in many books): "Either ABC or ABD is the greater. Suppose it is ABC, so that BD lies in the angle ABC. If possible, let the circle through A, B and C not pass through D. Then since BD lies within the angle ABC, the circle must cut BD or BD produced. . . ."

This makes use of a fact about the circle which does not follow from the definition and is not inherent in any previous theorem. If it is called "intuitive", so are other facts about the circle which are carefully proved—this particular converse, for example. H. E. Piggott.

CONFERENCE OF EDUCATIONAL ASSOCIATIONS, 1937.

The Council has decided to apply for affiliation to the above Conference and the necessary steps are being taken. This will entitle members to attend a number of meetings of other associations interested in educational matters, and also to take part in various excursions and other activities. The Conference will take place during the same week as the Annual Meeting of the Association and a pamphlet in connection with it will be sent to members at the same time as the preliminary notice of the Annual Meeting.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. The names of those sending the questions will not be published.

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